APPLIED MATHS - 3 SOLUTIONS OF QUESTION PAPER

CBCGS DEC (2018)

1. a) Find Laplace transform of f (t) = $\int_0^t u\ e^{-3u} \sin u\ \delta u$ **(5)**

Sol. We have L $(sin u)$ = $\frac{1}{s^2+1^2}$

$\therefore$ L (u $sin u$) = (-1) $\frac{\partial}{\partial s}\left[\frac{1}{s^2+1^2}\right]$

$= -1\left[\frac{(s^2+1^2)(0)-(2s)}{(s^2+1^2)^2}\right]$

$= -1\left[\frac{-(2s)}{(s^2+1^2)^2}\right]$

$= \left[\frac{(2s)}{(s^2+1^2)^2}\right]$

$\therefore$ L $[u\ e^{-3u} \sin u\ \delta u\ ] = \left[\frac{2(s+3)}{((s+3)^2+1^2)^2}\right]$

$\therefore$ L $\left[\int_0^t u\ e^{-3u} \sin u\ \delta u\ \right] = \frac{2}{s}\left[\frac{(s+3)}{((s+3)^2+1^2)^2}\right]$

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b) Show that the set of functions {cos nx , n = 1,2,3 …} is orthogonal on (0, 2π). **(5)**

Sol. We have f_n(x) = cos nx.

$\therefore \int_0^{2\pi} f_m(x)\cdot f_n(x)\,dx = \int_0^{2\pi} \cos mx \cdot \cos nx\ dx$

$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x \cdot \cos(m-n)x\ ]\,dx$

$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin\,(m-n)x}{m-n}\right]_0^{2\pi}$

$\therefore$ Now , two cases arise .

Case 1 : When m ≠ n, then $\int_0^{2\pi} f_m(x)\cdot f_n\ (x)\ dx = 0$

Case 2 : When m = n, then $\int_0^{2\pi} f_n(x) \cdot f_n(x)\, dx = \int_0^{2\pi} cos^2\, nx\, dx$

$\therefore \int_0^{2\pi} [\, f_n(x)]^2\, dx = \int_0^{2\pi} \left(\frac{1+cos2nx}{2}\right) dx = \frac{1}{2}\left[x + \frac{sin2nx}{2n}\right]_0^{2\pi} = \pi \neq 0$

Since , $\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, \; if\ m \neq n \\ \neq 0, if\ m = n \end{cases}$

the given set of functions is orthogonal on [0, 2π].

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c) Does there exist an analytic function whose real part is u = k (1+ cosϴ) ? Give justification **(5)**

Sol. Since , u = k (1+ cos θ)

$\therefore$ u = k + k cos θ,

$\frac{\partial u}{\partial r}$ = 0 and $\frac{\partial u}{\partial \theta}$ = - k sinθ.

But by Cauchy's Riemann equations in polar co-ordinates .

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$\therefore \quad \frac{\partial v}{\partial \theta} = 0\,, \frac{\partial v}{\partial r} = -\frac{1}{r}(-\,k\,\sin\theta)$

Integrating the first equation partially w.r.t. θ,

v = f(r) where f(r) is an arbitrary function.

(If v = f(r) then $\frac{\partial V}{\partial \theta} = 0$)

$\therefore \frac{\partial v}{\partial r} = f'(r) = \frac{k\, sin\theta}{r}$ $\qquad \therefore$ v = k $sin\theta$ log r + c

Hence , the analytic function is

f(z) = u +iv = k (1+ cos ϴ) + i k $sin\theta$ log r + c

$\therefore$ **Yes , this type of analytic function whose real part is u = k (1+ cos ϴ) exist.**

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d) The equations of lines of regression are x + 6y = 6 and 3x + 2y = 10 . **(5)**

Find (i) means of x and y , (ii) coefficient of correlation between x and y .

Sol. Given data : x + 6y = 6 .... (1)

3x + 2y = 10 .... (2)

(i) Adding equation (1) and (2) , we get

x = 3 and y = 0.5

(ii) x + 6y = 6

Slope 1 = $-\frac{1}{6}$

Slope 1 = byx

$\therefore$ $-\frac{1}{6}$ = byx

3x + 2y = 10

Slope 2 = $-\frac{3}{2}$

Slope 2 = $\frac{1}{bxy}$

$\therefore$ bxy = $-\frac{2}{3}$

r = $\sqrt{\boldsymbol{bxy\ byx}} = \sqrt{\left(\frac{-1}{6}\right).\left(\frac{-2}{3}\right)} = \frac{-1}{3}$ **.**

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2. a) Evaluate $\int_0^{\infty} e^{-t}\frac{sin^2 t}{t}\,dt$ **.** **(6)**

Sol. Step 1 : Comparing with standard formula $\int_0^{\infty} e^{-st} f(t)\,dt$ **.**

$\therefore s = 1$ & f(t) = $\frac{sin^2 t}{t}$

Step 2 : L [f(t)] = L $\left[\frac{sin^2 t}{t}\right]$

L $[sin^2 t]$ = L $\left[\frac{1-cos2t}{2}\right]$

= $\frac{1}{2}$ L$[1 - cos2t]$

= $\frac{1}{2}$ L$\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

By Division of t ,

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{1}{2} \times \frac{2s}{s^2+2^2}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+2^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+2^2) - \log s\right]$$

Step 3 : Substitute s = 1

$$= \frac{1}{2}\left[\frac{1}{2}\log(1+2^2) - \log 1\right]$$

$$= \frac{1}{4}\log 5 - 0$$

$$= \frac{1}{4}\log 5$$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log 5$$

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b) Find the image of the triangle bounded by lines $x = 0, y = 0\ ,\ x + y = 1$ in the z – plane under the transformation $w = e^{i\pi/4} z$. **(6)**

Sol.

Step 1 : Draw region in z – plane

$x = 0, y = 0\ ,\ x + y = 1$ → $x = 0, y = 1$, **(0,1)**

$x = 1, y = 0$, **(1.0)**

Step 2 : $w = e^{i\pi/4} z$

$$= e^{i\pi/4}(x + i\, y)$$

$$u + iv = e^{i\pi/4}(x + i\, y)$$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\ (x + iy)$$

$$u + iv = (1 + i)(x + i\, y)$$

$u + iv = x + iy + ix - y$

$\therefore u = x - y \; ; v = x + y$

Step 3 : $u = x - y$

$x = 0$;	$u = -y, v = y$
$x = 1$;	$u = 1 - y \, , v = 1 + y$
$y = 0$;	$u = x \, , v = x$
$y = 1$;	$u = x - 1, v = x + 1$

Now , using these points plot the graph on x and y axis to get the image of the triangle bounded by lines.

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c) Obtain Fourier series of $f(x) = x^2$ in (0,2π) . Hence , deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$ (8)

Sol . Let $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \;\; in \; (0, 2\pi)$

Then , $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x^2 dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx \, dx$$

(By the generalised rule of integration by parts)

$$a_n = \frac{1}{\pi}\left[x^2 \left(\frac{\sin nx}{n}\right) - 2x \left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + \frac{2}{n^3}\right\} - \left\{\frac{2}{n^3}\right\}\right]$$

$$= \frac{4\pi}{n}$$

Putting these values in (1),

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\therefore\ x^2 = \frac{4\pi^2}{3} + 4\left[\frac{cosx}{1^2} + \frac{cos2x}{2^2} + \frac{cos3x}{3^2} + \cdots\right]$$
$$- 4\pi\left[\frac{sinx}{1} + \frac{sin2x}{2} + \frac{sin3x}{3} + \cdots\right]$$

Now put x = π ,

$$\therefore \pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\therefore \frac{\pi^2}{12} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

Now , the given function , y = x^2 is parabola with vertex at the origin and opening upwards . Further when x = 0, y = 0 and when x = 2π , y = $4\pi^2$.

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3. a) Find the inverse Laplace transform of $f(s)$= $\frac{s}{(s^2+4)^2}$ (6)

Sol. Step 1 :

Let $f(s)$= $f_1(s) \cdot f_2(s)$

$\therefore\ f_1(s) = \frac{s}{s^2+2^2}$, $f_2(s) = \frac{1}{s^2+2^2}$

Step 2 :

$L^{-1}[f_1(s)] = \frac{s}{s^2+2^2} = \cos 2t \rightarrow f_1(t)$

$L^{-2}[f_2(s)] = \frac{1}{s^2+2^2} = \frac{\sin 2t}{2} \rightarrow f_2(t)$

Step 3 :

$f_1(u) = \cos 2u$

$f_2(t-u) = \frac{1}{2}\sin 2(t-u)$

Step 4 :

$\int_0^t f_1(u) \cdot f_2(t-u)\, du$

$= \frac{1}{2}\int_0^t \cos 2u \cdot \sin(2t-2u)\ du$

$= \frac{1}{2\cdot 2}\int_0^t 2\cos 2u \cdot \sin(2t-2u)\ du$

$= \frac{1}{4}\int_0^t \sin 2t - \sin(4u-2t)\ du$

$= \frac{1}{4}\left[\sin 2t \cdot u + \frac{cos(4u-2t)}{4}\right]_0^t$

$= \frac{1}{4}\left[\left(\text{t}\sin 2t + \frac{\cos(2t)}{4}\right) - \left(0 + \frac{\cos(-2t)}{4}\right)\right]$

$= \frac{tsin2t}{4}$

$L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{tsin2t}{4}$

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b) Solve $\frac{\partial^2 u}{\partial x^2} - 100\ \frac{\partial u}{\partial t} = 0$, with $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = x\,(1 - x)$ taking h=0.1 for three time steps up to t = 1.5 by Bender – Schmidt method. **(6)**

Sol. $\frac{\partial^2 u}{\partial x^2} - 100\ \frac{\partial u}{\partial t} = 0$

$u(0,t) = 0$

$u(1,t) = 0$

$u(x,0) = x\,(1-x)$

h = 0.1 & t = 1.5

$k = \frac{ah^2}{2} = \frac{100(0.1)^2}{2}$ = 0.5

$x \rightarrow$ / $t \downarrow$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	**0**	**0.09**	**0.16**	**0.21**	**0.24**	**0.25**	**0.24**	**0.21**	**0.16**	**0.09**	**0**
0.5	**0**	**0.08**	**0.15**	**0.2**	**0.23**	**0.24**	**0.23**	**0.2**	**0.15**	**0.08**	**0**
1	**0**	**0.075**	**0.14**	**0.19**	**0.22**	**0.23**	**0.22**	**0.19**	**0.14**	**0.075**	**0**
1.5	**0**	**0.07**	**0.1325**	**0.18**	**0.21**	**0.22**	**0.21**	**0.18**	**0.1325**	**0.07**	**0**

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c) Using Residue theorem , evaluate **(8)**

(i) $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$

Sol. Now , put z = $e^{i\theta}$

$\therefore dz = i\, e^{i\theta} \cdot d\theta$

$\therefore dz = i\, z\, d\theta$

$\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+(1/z)}{2} = \frac{z^2+1}{2z}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-4cos\theta} = \int_C \frac{1}{5-4\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-4cos\theta} = \int_C \frac{2z}{2(-2z^2+5z-2)} \cdot \frac{dz}{iz}$$

$$= -\int_C \frac{1}{(2z^2+5z-2)} \cdot \frac{dz}{i}$$

Now the poles are given by $2z^2 + 5z - 2 = 0$

$\therefore (2z-1)(z-2) = 0$

$\therefore z = \frac{1}{2} and\ z = 2\ .$

The pole $z = \frac{1}{2}$ lies inside the unit circle and $z = 2$ lies outside it .

Now , Residue of f(z) (at z= 1/2) = $\lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) \cdot \frac{(-1)}{i\cdot 2[z-(1/2)](z-2)}$

= $\lim_{z\to\frac{1}{2}} \frac{(-1)}{i\cdot 2[\left(\frac{1}{2}-2\right)]}$

= $\frac{1}{3i}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5-4cos\theta} = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$$

Real part of residue = $\frac{2\pi}{3}$

(ii) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

Sol. 1) Region lies in upper half of circle .

2) Replace x by z

3) Poles : $(x^2+1)^2 = 0$

$$(z^2+1)^2 = 0$$

$$(z^2-i^2)^2 = 0$$

$$[(z+i)(z-i)]^2 = 0$$

$$(z+i)^2 = 0 \ ; (z-i)^2 = 0$$

$$z = \ -i\,,-i \quad ; \quad z = i\,,i$$

4) Residue $f(z) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} f(z-z_0)^2\ (z-z_0)$

$$= \frac{1}{1!} \lim_{z\to i} \frac{d}{dz}\ (z-i)^2\ \frac{1}{(z+i)^2(z-i)^2}$$

$$= \lim_{z\to i} \frac{d}{dz}\ (z+i)^{-2}$$

$$= \ -2(z+i)^{-3}$$

$$= \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{(i+i)^3}$$

$$= \frac{-2}{(2i)^3}$$

$$= \frac{-2}{-8i}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{2}{8i}.$$

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4. a) Solve by Crank – Nicholson simplified formula $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0.$

$u(0,t) = 0$ **,** $u(5,t) = 100$ **,** $u(x,0) = 20$ **taking h = 1 for one time step. (6)**

Sol. $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0.$

$u(0,t) = 0$

$u(5,t) = 100$

$u(x,0) = 20$

∴ Taking h = 1 for one time step,

∴ a = 1 & h = 1

$\boldsymbol{k = ah^2}$ **= 1.**

x / t	0	1	2	3	4	5
0	**0**	**20**	**20**	**20**	**20**	**100**
1	**0**	**9.80**	**2.19**	**30.72**	**59.92**	**100**

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b) Obtain the Taylor's and Laurent's series which represent the function $\boldsymbol{f(z) = \frac{z-1}{z^2-2z-3}}$ **in the regions** **(6)**

i) $\boldsymbol{|z| < 1}$

ii) $\boldsymbol{1 < |z| < 3}$

Sol. Let $f(z) = \frac{z-1}{z^2-2z-3} == \frac{a}{z+1} + \frac{b}{z-3}$

$\therefore z - 1 = a(z-3) + b(z+1)$

Putting $\boldsymbol{z = -1}$, -2 = -4a $\therefore a = \frac{1}{2}$

Putting $\boldsymbol{z = 3}$**,** 2 = b ·4 $\therefore b = \frac{1}{2}$

$\therefore \frac{z-1}{z^2-2z-3} = \frac{1/2}{z+1} + \frac{1/2}{z-3}$

Hence , $\boldsymbol{f(z)}$ **is not analytic at** $\boldsymbol{z = -1}$ **and** $\boldsymbol{z = 3}$.

$\therefore\ f(z)$ is analytic in i) $|z| < 1$ ii) $1 < |z| < 3$

(i) $f(z) = \frac{1/2}{z+1} + \frac{1/2}{z-3} = \frac{1/2}{1+z} + \frac{1/2}{(-3)} \cdot \frac{1}{1-(\frac{z}{3})}$

$$= \frac{1}{2} \cdot [1+z]^{-1} - \frac{1}{6}\left(1 - \frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2} \cdot [1 - z + z^2 - z^3 + \ \dots] - \frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \ \dots\right]$$

$$= \frac{1}{3} - \frac{5}{9}z + \frac{13}{27}z^2 + \ \dots$$

This is the required Taylor's Series.

(ii) When $1 < |z| < 3$, we get $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$.

$$\therefore\ f(z) = \frac{1}{2} \cdot \frac{1}{1+z} + \frac{1}{2} \cdot \frac{1}{z-3}$$

$$= \frac{1}{2z} \cdot \frac{1}{1+\left(\frac{1}{z}\right)} + \frac{1}{2} \cdot \frac{1}{(-3)} \cdot \frac{1}{1-\left(\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left[1 + \frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1 - \frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2z} \cdot \left[1 - \frac{1}{z} + \frac{1}{z}^2 - \frac{1}{z}^3 + \ \dots\right] - \frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \ \dots\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \ \dots\right] - \frac{1}{6}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \ \dots\right]$$

This is the required Laurent's Series.

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c) Solve $(D^2 + 4D + 8)y = 1$ with $y(0) = 0$ and $y'(0) = 1$,

where $D \equiv \frac{d}{dt}$ **(8)**

Sol. Let $\bar{y}$ be the Laplace Transform of y i.e. let L(y) = $\bar{y}$.

Taking Laplace transform of the both sides ,

L(y'') + 4L(y') + 8L(y) = L(1)

Now L(y') = s$\bar{y}$ - y(0) = s$\bar{y}$

L(y'') = $s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$ and L(1) = $\frac{1}{s}$

$\therefore$ **The equation (1) becomes**

$s^2\bar{y} - 1 + 4s\,\bar{y} + 8\,\bar{y} = \frac{1}{s}$

$$\bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\bar{y} = \frac{s+1}{s(s^2 + 4s + 8)}$$

$$\therefore y = L^{-1}(\bar{y}) = L^{-1}\frac{s+1}{s(s^2 + 4s + 8)}$$

We obtain the L^{-1} by the partial fractions

$$\therefore\ y = L^{-1}\left[\frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2+4s+8} + \frac{1}{2}\cdot\frac{1}{s^2+4s+8}\right]$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\frac{(s+2)-2}{(s+2)^2+2^2} + \frac{1}{2}L^{-1}\frac{1}{(s+2)^2+2^2}$$

$$= \frac{1}{8}\cdot 1 - \frac{1}{8}e^{-2t}L^{-1}\frac{s}{(s)^2+2^2} + \frac{6}{8}e^{-2t}L^{-1}\frac{s}{(s)^2+2^2}$$

$$\therefore\ y = \frac{1}{8} - \frac{1}{8}e^{-2t}cos2t + \frac{3}{8}e^{-2t}\ sin2t.$$

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5 . a) Find an analytic function $f(z) = u + iv$, if

$\boldsymbol{u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\ sin y\}}$ **(6)**

Sol. $f(z) = u + iv$

$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy\ siny\}$

1) $u = e^{-x}(x^2 - y^2)\cos y + e^{-x}\ 2xy\ siny$

$u_x = cosy\,[e^{-x}(2x) + (x^2 - y^2)e^{-x}] + siny\,(e^{-x}(2y) + 2xy\,e^{-x})$

$u_y = e^{-x}\,[(x^2 - y^2)(-siny) - cosy(-2y)] + e^{-x}\,[2xy\ cosy - siny\ 2x].$

2) $f'(z) = u_x - u_y$

$= [2xe^{-x}cosy + e^{-x}cosy(x^2 - y^2) + 2ye^{-x}\ siny + 2xye^{-x}\ siny] - i[-e^{-x}(x^2 - y^2)siny + 2y\ cosy + 2xye^{-x}\ cosy - 2xe^{-x}siny]$

$x = z; y = 0$

3) $f'(z) = (2ze^{-z} + e^{-z}z^2)$

$$f(z) = 2\left[\frac{ze^{-z}}{(-1)} - \frac{e^{-z}}{(-1)^2}\right] + \frac{z^2e^{-z}}{(-1)} - \frac{2ze^{-z}}{(-1)^2} + \frac{2e^{-z}}{(-1)^3}$$

$$= -4ze^{-z} - 3e^{-z} - z^2e^{-z}$$

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b) Find the Laplace Transform of **(6)**

$$\boldsymbol{f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}}$$ **and $f(t+2) = f(t)$ for t > 0.**

Sol. $f(t)$ is periodic & t = 2

$$L[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt + \int_1^2 e^{-st}(0)dt\right]$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}tdt$$

$$= \frac{1}{1-e^{-2s}}\left[t\frac{e^{-st}}{-s} - 1\cdot\frac{e^{-st}}{(-s)^2}\right]_0^1$$

$$= \frac{1}{1-e^{-2s}}\left[\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{(s)^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{(s)^2}\right) + \frac{1}{s^2}\right]$$

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c) Obtain half range Fourier cosine series of $f(x)$ = x,$0 < x < 2$. Using Parseval's identity , deduce that – **(8)**

$$\boldsymbol{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots}$$

Sol. Let of $f(x)$ = $\boldsymbol{a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)}$.

Here , $\boldsymbol{l} = \mathbf{2}$.

$$\therefore \; a_0 = \frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$= \tfrac{2}{l}\int_0^l f(x)\boldsymbol{cos}\left(\frac{\boldsymbol{n\pi x}}{\boldsymbol{l}}\right)\,dx = \tfrac{2}{2}\int_0^2 x\,\boldsymbol{cos}\left(\frac{\boldsymbol{n\pi x}}{\boldsymbol{l}}\right)\,dx$$

$$= \left[x\,\frac{sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{2^2}}\cdot 1\right]_0^2$$

$$\therefore \; a_0 = \left[2\cdot(0) + \frac{cos(n\pi x)}{\frac{n^2\pi^2}{2^2}} - 0 - \frac{1}{\frac{n^2\pi^2}{2^2}}\cdot\right] = \frac{[(-1)^n - 1]}{\frac{n^2\pi^2}{2^2}}$$

$$= \begin{cases} -4\cdot\frac{2}{n^2\pi^2} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$$

$$\therefore x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}cos\frac{3\pi x}{2} + \cdots\right]$$

BY Parseval's identity ,

$$\frac{1}{l}\int_0^l [f(x)]^2 dx = \frac{1}{2}[2{a_0}^2 + {a_1}^2 + {a_2}^2 + \cdots]$$

$$\therefore L.H.S = \int_0^2 x^2 dx\,\tfrac{1}{2}\left[\tfrac{x^3}{3}\right]_0^2 = \tfrac{4}{3}$$

$$\therefore \; \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right\}\right]$$

$$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right\}$$

$$\boldsymbol{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots}$$

Hence proved ,

6. a) If $f(a) = \int_c \frac{4z^2+z+4}{z-a}\, dz$ where C is the ellipse $4x^2 + 9y^2 = 36$. (6)

Find i) $f(4)$

ii) $f'(-1)$

iii) $f''(-i)$

Sol. $\therefore 4x^2 + 9y^2 = 36.$

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

$$f(a) = \int_c \frac{4z^2+z+4}{z-a}\, dz = 2\pi i \emptyset(z_0)$$

i) $z_0 = 4$

$\emptyset(z) = 4z^2 + z + 4$

$\emptyset(z_0) = \emptyset(4) = 4(4)^2 + 4 + 4 = 72$

$f(4) = 2\pi i \emptyset(4) = 2\pi i \times 72 = 144\pi i$

ii) $\emptyset'(z) = 8z + 1$

$\emptyset'(-1) = 8(-1) + 1 = -7$

$f'(-1) = 2\pi i \emptyset'(-1) = 2\pi i(-7) = -14\pi i$

iii) $\emptyset''(z) = 8$

$\therefore \emptyset''(i) = 8$

$f''(i) = 2\pi i \emptyset''(i) = 2\pi i(8) = 16\pi i$

b) Use least square regression to fit a straight line to the following data,   (6)

X	5	10	15	20	25	30	35	40	45	50
Y	**17**	**24**	**31**	**33**	**37**	**37**	**40**	**40**	**42**	**41**

Sol.

X	Y	xy	x^2
5	17	85	25
10	24	240	100
15	31	465	225
20	33	660	400
25	37	925	625
30	37	1110	900
35	40	1400	1225
40	40	1600	1600
45	42	1890	2025
50	41	2050	2500
275	**342**	**10425**	**9625**

$$\sum y = n\,a + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Substituting the values ,

342 = 10 a   + 275 b

10425 = 275 a + 9625 b

$\therefore a = 20.6 \,\&\, b = 0.495$

General equation for the straight line ,

y = a + b x = 20.6 + 0.495 x

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6 .c) A string is stretched and fastened to two points distance l apart. Motion is started by displacing the string in the form $y = asin\left(\frac{\pi x}{l}\right)$ from which it is

released at time t = 0. If the vibration of a string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, show that the displacement of a point at a distance x from one end at time t is given by $y_{(x,t)} = a\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$. **(8)**

Sol. The vibration of a string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ....(1)

Since the vibration of a string is periodic the solutions of (1) is of the form

$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct)$ ....(2)

Given initial boundary condition are :

(i) When x = 0, y = 0 for all t i.e. one end A of the string remains fixed throughout the motion : We get , from (2)

$0 = (c_1 + 0)(c_3 \cos mct + c_4 \sin mct)$ $\therefore\ c_1 = 0$

$\therefore\ y = c_2 \sin mx\,(c_3 \cos mct + c_4 \sin mct)$

(ii) Now , $\frac{\partial y}{\partial t} = 0$ when t = 0 i.e. when initially the string is steady (initially velocity is zero)

From (3) $\frac{\partial y}{\partial t} = c_2 \sin mx\,(c_3(-mc)\sin mct + c_4(mc)\cos mct)$

Putting t = 0 , $\frac{\partial y}{\partial t} = 0$,

$\therefore\ 0 = c_2 \sin mx\,(c_4\, mc)$

$\therefore\ c_2 c_4\, mc = 0$.

If $c_2 = 0$ then (3) will give a trivial solution y=0

Thus from (3), we get

y = $c_2 c_3 \sin mx \cos mct$

= $c_5 \sin mx \cos mct$ (where $c_2 c_3 = c_5$) ....(4)

(iii) Now y = 0 when x = l for all t i.e. the other end of the string is fixed and the length of the string is l,

$\therefore$ From (4), we get ,

$\therefore$ 0 = $c_5 \sin mx \cos mct$ ....(5)