MUMBAI UNIVERSITY

SEMESTER – 2

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 a) Evaluate $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx$. **[3]**

ANS: $I = \int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx.$

Put $x^3 = t$

$\therefore x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{\frac{-2}{3}}$

$\therefore I = \int_0^\infty e^{-t} . t^{-\frac{1}{6}} . \frac{1}{3} . t^{\frac{-2}{3}} dt$

$\therefore I = \int_0^\infty e^{-t} . t^{-\frac{5}{6}} dt$

$\therefore \boldsymbol{I = \frac{1}{3} \left| \overline{\frac{1}{6}} \right.}$.

b) Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from y = 1 to y = 2. [3]

ANS: We have $x = \frac{y^3}{3} + \frac{1}{4y}$

Diff w.r.t. y, we get

$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^2-\frac{1}{4y^2}\right)^2 = y^4+\frac{1}{2}+\frac{1}{16y^4} = \left(y^2+\frac{1}{4y^2}\right)^2$$

We know that,

$$s = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$s = \int_1^2 \sqrt{\left(y^2+\frac{1}{4y^2}\right)^2}\, dy$$

$$s = \left[\frac{y^3}{3}-\frac{1}{4y}\right]_1^2$$

$$s = \frac{8}{3}-\frac{1}{8}-\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$\boldsymbol{s = \frac{59}{24}}$$

c) Solve $(D^2+D)y = e^{4x}$. **[3]**

ANS: For auxiliary equation,

$D^2 + D = 0$

Solving we get,

D = -1, 0.

$\therefore$C.F. = $C_1e^{-x} + C_2e^{0x}$

$\therefore C.F = C_1e^{-x} + C_2$

For P.I.,

$$y = \frac{e^{4x}}{D^2+D}$$

Now, put D = 4

$$\therefore y = \frac{e^{4x}}{4^2+4} = \frac{e^{4x}}{20}$$

$\therefore$ The complete solution is,

$$\boldsymbol{y = C_1e^{-x} + C_2 + \frac{e^{4x}}{20}}.$$

d) Evaluate $\int_0^1\int_{x^2}^{x} xy(x+y)\,dy\,dx$. **[3]**

ANS: We have,

$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)dydx.$$

$$I = \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$I = \left[\frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$I = \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$\boldsymbol{I = \frac{3}{56}}$$

e) Solve $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0$. **[4]**

ANS: Given, $(4x + 3y - 4)dx + (3x - 7y - 3)dy = 0$.

∴ M = $(4x + 3y - 4)$ and N = $(3x - 7y - 3)$

Differentiating M by y and N by x, we get,

$\frac{dM}{dy} = 3$ And $\frac{dN}{dx} = 3$

$\therefore \frac{dM}{dy} = \frac{dN}{dx}$

∴The given equations are exact.

For solution,

$\int M\, dx = \int (4x + 3y - 4)\, dx$

$\int M\, dx = 2x^2 + 3xy - 4x$

$\int (Term\ is\ N\ free\ from\ x) = \int -7y - 3\ dy$

$= \frac{-7y^2}{2} - 3y$

∴ The final solution is,

$$2x^2 + 3xy - 4x - \frac{7y^2}{2} - 3y = c$$

$$4x^2 + 6xy - 8x - 7y^2 - 6y = c$$

f) Solve $\frac{dy}{dx} = 1 + xy$ with initial condition $x_0 = 0,\ y_0 = 0.2$ By Taylors series method. Find the approximate value of y for x = 0.4(step size = 0.4).

ANS: The Taylor series is given by,

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0'''..... \qquad .... (1)$$

With $x_0 = 0, y_0 = 0.2, x = 0.4$

Now, y' = 1 + xy $\quad \therefore y_0' = 1$

y'' = y + xy' $\quad \therefore y_0'' = y_0 = 0.2$

y''' = y' + y' + xy''

= 2y' + xy'' $\quad \therefore y_0''' = 2y_0' = 2$

y'''' = 2y'' + y'' + xy''' $\quad \therefore y_0'''' = 3y_0'' + xy_0''' = 0.6$

Putting these values in equation 1, we get

$$y = 0.2 + (0.4)1 + + \frac{(0.4)^2}{2!}0.2 + \frac{(0.4)^3}{3!}.2 + \frac{(0.4)^4}{4!}.(0.6) + \cdots$$

$$y = 0.2 + 0.4 + 0.016 + 0.02133 + 0.00064$$

$$y = 0.63797.$$

Q.2 a) Solve $\frac{d^2y}{dx^2} - 16y = x^2e^{3x} + e^{2x} - \cos 3x + 2^x$. [6]

ANS: The auxiliary equation is $D^2 - 16 = 0$

$\therefore D = 4, -4$

$\therefore$ The C.F. is $y = C_1e^{4x} + C_2e^{-4x}$

Now, to find P.I.,

P.I. = $\frac{1}{D^2-16}(x^2e^{3x} + e^{2x} - \cos 3x + 2^x)$

Now, $\frac{1}{D^2-16}x^2e^{3x} = e^{3x}.\frac{1}{(D+3)^2-16}.x^2$

$$= e^{3x}.\frac{1}{D^2+6D+9-16}.x^2 = e^{3x}.\frac{1}{D^2+6D-7}.x^2$$

$$= -\frac{e^{3x}}{7} \cdot \frac{1}{\left(1-\frac{D^2+6D}{7}\right)} \cdot x^2$$

$$= -\frac{e^{3x}}{7} \cdot \left(1 - \frac{D^2+6D}{7}\right)^{-1} \cdot x^2$$

$$= -\frac{e^{3x}}{7}\left(1 + \frac{D^2+6D}{7} + \frac{D^4+6D^3+36D^2}{49} + \cdots\right) \cdot x^2$$

$$= -\frac{e^{3x}}{7}\left(x^2 + \frac{12x+2}{7} + \frac{72}{49}\right) = -\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right)$$

$$\therefore \frac{1}{D^2-16} \cdot e^{2x} = e^{2x}\frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{2^2-16} = e^{2x} \cdot \frac{1}{12}$$

$$\therefore \frac{1}{D^2-16} \cdot \cos 3x = \frac{\cos 3x}{-9-16} = \frac{\cos 3x}{-25}$$

$$\therefore \frac{1}{D^2-16} \cdot 2^x = \frac{1}{D^2-16} \cdot e^{x\log 2} = \frac{e^{x\log 2}}{(\log 2)^2-16} \cdot = \frac{2^x}{(\log 2)^2-16}.$$

$\therefore$P.I. = $-\frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}.$

$\therefore$ The complete equation is,

$$\boldsymbol{y = C_1e^{4x} + C_2e^{-4x} - \frac{e^{3x}}{7}\left(x^2 + \frac{12x}{7} + \frac{86}{49}\right) + e^{2x} \cdot \frac{1}{12} + \frac{\cos 3x}{25} + \frac{2^x}{(\log 2)^2-16}}.$$

b) Show that $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx = \pi \sin^{-1} a \quad 0 \leq a \leq 1$. [6]

ANS: Let I (a) be the given integral. By the rule of differentiation under the integral sign.

$$\frac{dI}{da} = \int_0^{\pi} \frac{df}{da} dx = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{\cos x}{1+a\cos x} dx = \int_0^{\pi} \frac{dx}{1+a\cos x}$$

Put $t = tan\frac{x}{2}, dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

When x = 0, t = 0;

When x =π, t = $tan\frac{\pi}{2}$=∞

$$\therefore \frac{dI}{da} = \int_0^\infty \frac{1}{1+a.(\frac{1-t^2}{1+t^2})} \cdot \frac{2\,dt}{1+t^2}$$

$$\frac{dI}{da} = \int_0^\infty \frac{2\,dt}{(1+t^2)+a(1-t^2)}.$$

$$\frac{dI}{da} = \int_0^\infty \frac{2\,dt}{(1+a)+(1-a)t^2}.$$

$$\frac{dI}{da} = \frac{1}{1-a}\int_0^\infty \frac{2\,dt}{\left[\frac{1+a}{1-a}\right]+t^2}$$

$$\frac{dI}{da} = \frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}.\left[\tan^{-1}\sqrt{\frac{1-a}{1+a}}\right]_0^\infty$$

$$\frac{dI}{da} = \frac{2}{\sqrt{1-a^2}}.\frac{\pi}{2}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}.$$

Integrating both sides w.r.t. a, we get

$$I = \pi\ \sin^{-1} a + c$$

To find c, put a = 0

$$I(0) = \pi\ \sin^{-1} 0 + c, c = 0$$

$$\therefore I = \pi \sin^{-1} a$$

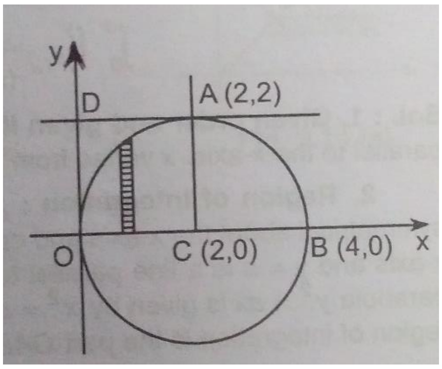
$$\boldsymbol{\therefore \int_0^\pi \frac{\log(1+a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a}$$

c) Change the order of integration and evaluate $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dxdy$. **[8]**

ANS: 1) Given order and given limits: Given order is: first w.r.t. x and then w.r.t y i.e., a strip parallel to the x-axis varies from x = $2-\sqrt{4-y^2}$ to x = $2-\sqrt{4-y^2}$. Y varies from y = 0 to y = 2.

2) Region of integration: x = $2-\sqrt{4-y^2}$ is the arc and x = $2+\sqrt{4-y^2}$ is the arc of the circle $(x-2)^2+y^2=4$with centre at (2, 0) and radius = 2 above the x-axis. y = 0 is the x-axis and y = 2 is the line parallel to the x-axis through A (2, 2). The region of integration is the semi-circle OAB above the x-axis. The points of intersection of the circle and the x-axis are O (0, 0) and B (4, 0).

3) Change of order of integration: To change the order, consider a strip parallel to the y-axis in the region of integration. On this strip y varies from y = 0 to y =$\sqrt{4-(x-2)^2}$ and then strip moves from x = 0 to x = 4.



$$I = \int_0^4 \int_0^{\sqrt{4-(x-2)^2}} dydx$$

$$I = \int_0^4 [y]_0^{\sqrt{4-(x-2)^2}} dx$$

$$I = \int_0^4 \sqrt{4-(x-2)^2}\, dx$$

$$I = \left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + 2\sin^{-1}\frac{x-2}{2}\right]_0^4$$

$$I = \left(2.\frac{\pi}{2}\right) - (-2.\frac{\pi}{2})$$

$$\therefore \boldsymbol{I = 2\pi}$$

Q.3 a) Evaluate $\iiint (x+y+z)\, dxdydz$ over the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x + y + z = 1. [6]

ANS:

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z)dzdydx$$

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dydz$$

$$I = \frac{1}{2}\int_{x=0}^{1} \int_{y=0}^{1-x} [1-(x+y)^2]dydx$$

$$I = \frac{1}{2}\int_{x=0}^{1}\left[y - \frac{(x+y)^2}{2}\right]_0^{1-x} dx$$

$$I = \frac{1}{2}\int_{x=0}^{1}[(1-x) - \frac{1}{3} + \frac{x^3}{3}]dx$$

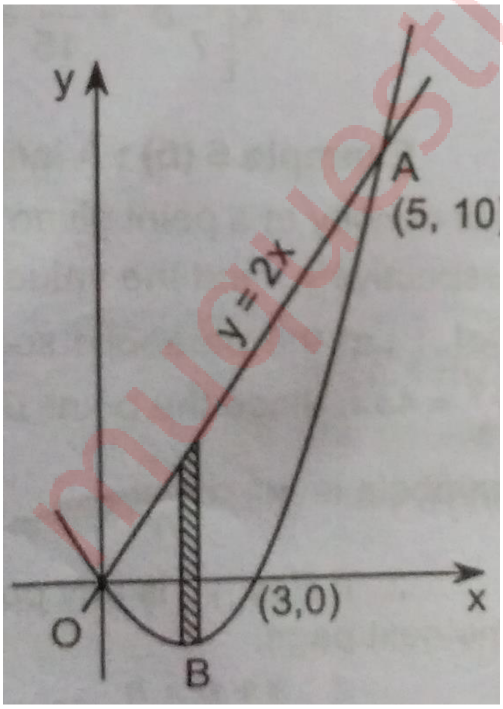
$$I = \frac{1}{2}\left[\frac{2x}{3} - \frac{x^2}{2} + \frac{x^4}{12}\right]_0^1$$

$$I = \frac{1}{2}.\frac{3}{12} = \frac{1}{8}$$

$$\therefore \boldsymbol{I = \frac{1}{8}}$$

b) Find the mass of lamina bounded by the curves $y = x^2 - 3x$ and $y = 2x$ if the density of the lamina at any point is given by $\frac{24}{25}xy$. [6]

ANS: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = (x - \frac{3}{2})^2$ is parabola intersecting the x-axis in x = 0 and x = 3. The line y = 2x intersects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at x = 0, x = 5. Therefore, points of intersection are (0,0) and (5,10). The surface density is ρ = (24/25)xy. Taking the elementary strip parallel to the y-axis, on the strip y varies from $y = x^2 - 3x$ to $y = 2x$ and then x varies from x = 0 to x = 5.



$\therefore$ Mass of lamina = $\int_0^5 \int_{x^2-3x}^{2x} \frac{24}{25} xydxdy$

$$= \frac{24}{25} \int_0^5 x \left[\frac{y^2}{2}\right]_{x^2-3x}^{2x} dx$$

$$= \frac{24}{50} \int_0^5 4x^3 - x(x^4 - 6x^3 + 9x^2)]dx$$

$$= \frac{24}{50} \int_0^5 -5x^3 + 6x^4 - x^5)]dx$$

$$= \frac{24}{50} \left[\frac{-x^6}{b} + \frac{6x^5}{5} - \frac{5x^4}{4}\right]_0^5$$

$$= \frac{24}{50}.5^4[-\frac{25}{6} + 6 - \frac{5}{4}]$$

$$= \frac{24}{50}.5^4.\frac{7}{12}$$

$\therefore$Mass of lamina = 175.

c) Solve $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x .\cos(\log x)}{x}$ **[8]**

ANS: Given that,

$$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 3y = \frac{\log x .\cos(\log x)}{x}$$

Putting z = log x and x = e^z, we get

$$[D(D-1) + 3D + 3]y = e^{-z}.z.\cos z$$

$$[D^2 + 2D + 3]y = e^{-z}.z.\cos z$$

$\therefore$The A.E. is $D^2 + 2D + 3 = 0$

$$\therefore D = \frac{-2\pm 2\sqrt{2}.i}{2} = -1 \pm \sqrt{2}.i$$

$\therefore$The C.F. is $y = e^{-z}(C_1 \cos\sqrt{2}z + C_2 \sin\sqrt{2}z)$

P.I. =$\frac{1}{D^2+2D+3} e^{-z}.z.\cos z$

$$= e^{-z}.\frac{1}{(D-1)^2+2(D-1)+3}.z.\cos z = e^{-z}.\frac{1}{D^2+2}.z.\cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2}.2D\right].\frac{1}{D^2+2}.\cos z$$

$$= e^{-z}\left[z - \frac{1}{D^2+2}.2D\right]\cos z = e^{-z}\left[zcos\ z + \frac{1}{D^2+2}.2\sin z\right]$$

$$= e^{-z}[z\cos z + 2\sin z]$$

The complete solution is,

y = C.F. + P.I.

$$y = e^{-z}\left(C_1\cos\sqrt{2}z + C_2\sin\sqrt{2}z\right) + e^{-z}[z\cos z + 2\sin z]$$

$$\boldsymbol{y = \frac{1}{x}\left(C_1\cos\sqrt{2}\log x + C_2\sin\sqrt{2}\log x\right) + \frac{1}{x}[\log x\cos\log x + 2\sin\log x]}$$

Q.4 a) Find by double integration the area bounded by the parabola $y^2 = 4x$ And $y = 2x - 4$ [6]

ANS: The parabola $y^2 = 4x$ and the line $y = 2x - 4$ intersect where $(2x - 4)^2 = 4x$

$\therefore 4x^2 - 16x + 16 = 4x$ $\therefore 4x^2 - 20x + 16 = 0$

$\therefore x^2 - 5x + 4 = 0$ $\therefore$ (x - 4) (x - 1)=0

$\therefore$ x = 1, 4.

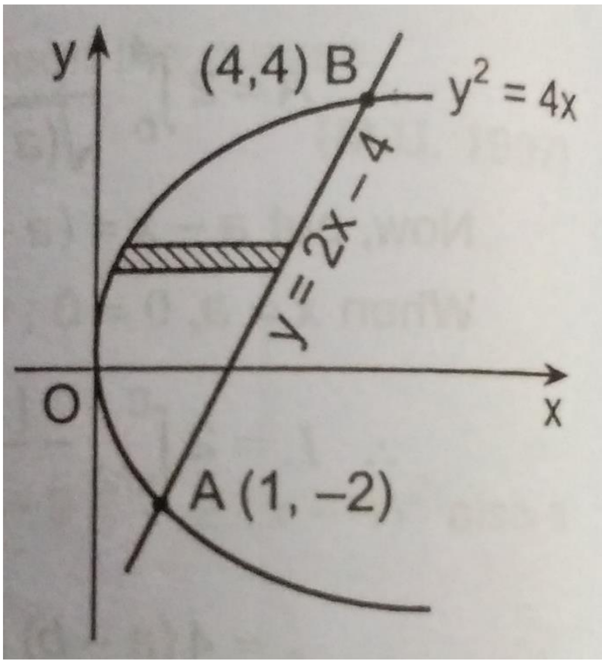
When x = 1, y = 2 – 4 = -2; and when x = 4, y = 8 - 4 = 4. Thus, the points of intersection are A (1, -2) and B (4, 4).

Now, consider a strip parallel to x-axis. On this strip x varies from x = $y^2/4$ to x = (y+4)/2. The strip then moves parallel to the x-axis from y = -2 to y = 4.

$$\therefore A = \int_{-2}^{4}\int_{y^2/4}^{(y+4)/2} dxdy = \int_{-2}^{4}[x]_{\frac{y^2}{4}}^{\frac{y+4}{2}} dy$$

$$= \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right)dy$$

$$= \frac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)dy$$



$$= \frac{1}{4}\left[y^2 + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right]$$

$$= \frac{1}{4}(60 - 24)$$

$\therefore \boldsymbol{A = 9}$

b) Solve $\frac{dy}{dx} + x\sin 2y = x^3 cos^2 y$ [6]

ANS: Given, $\frac{dy}{dx} + x\sin 2y = x^3 cos^2 y$

Dividing both sides by $\cos^2 x$,

$$sec^2 x\frac{dy}{dx} + x\, sec^2 x\sin 2y = x^3$$

$$sec^2 x\frac{dy}{dx} + 2x\tan y \ = x^3 \text{……………………. (1)}$$

Put tan y = v and differentiate w.r.t. x,

$$sec^2 x\frac{dy}{dx} = \frac{dv}{dx}$$

Hence, from (1), we get $\frac{dv}{dx} + 2v.x = x^3$

$\therefore\ P = 2x$ And Q = x^3

$\therefore \int P\,dx = \int 2x\,dx = x^2$

$\therefore$ I.F. = $e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ The solution is $v\,e^{x^2} = \int e^{x^2} x^3\,dx + c$

To find the integral put $x^2 = t,\ xdx = \frac{dt}{2}$.

$\therefore I = \int e^t.t.\frac{dt}{2} = \frac{1}{2}[te^t - \int e^t.dt]$.......... [By parts]

$\therefore I = \frac{1}{2}[te^t - e^t] = \frac{1}{2}e^t(t-1) = \frac{1}{2}e^{x^2}(x^2-1)$

$\therefore$ The solution is $v\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

$$\therefore \tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$$

c) Solve $\frac{dy}{dx} = x^3 + y$ with initial conditions $y(0) = 2$ at $x = 0.2$ in step of

h = 0.1 by Runge Kutta method of Fourth order. [8]

ANS: Given that, $\frac{dy}{dx} = x^3 + y$

$f(x,y) = x^3 + y, x_0 = 0, y_0 = 2\ and\ h = 0.1$

$\therefore k_1 = hf(x_0, y_0) = 0.1(0+2) = 0.2$

$\therefore k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2}{2}] = 0.2100$

$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.2105$

$\therefore k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right) = 0.1[(\frac{0.1}{2})^3 + 2 + \frac{0.2100}{2}] = 0.23105$

$\therefore k = \frac{k_1+2k_2+2k_3+k_4}{6} = \frac{0.2+2(0.21)+2(0.2105)+0.23105}{6}$

$$\therefore k = 0.2120$$

Q.5 a) Evaluate $\int_0^1 x^5 \sin^{-1} x\, dx$ and find the value of $\beta(\frac{9}{2},\frac{1}{2})$. **[6]**

ANS: $I = \int_0^1 x^5 \sin^{-1} x\, dx$

Put $\sin^{-1} x = t$ ∴x = sin t $dx = \cos t\, dt$

When x = 0, t = 0 when x = 1, t = $\pi/2$

$$I = \int_0^{\pi/2} \sin^5 t.t.\cos t\, dt = \int_0^{\pi/2} t(\sin^5 t.\cos t)\, dt$$

Integrating by parts,

$$I = \left[t.\frac{sin^6 x}{6}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{sin^6 x}{6}.1.dt$$

$$I = \left(\frac{\pi}{2}.\frac{1}{6} - 0\right) - \frac{1}{6}.\frac{5.3.1}{6.4.2}.\frac{\pi}{2}$$

$$I = \frac{\pi}{12} - \frac{5\pi}{192}$$

$$\therefore \boldsymbol{I = \frac{11\pi}{192}}$$

$$\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{\left\lceil\frac{9}{2}\left\lceil\frac{1}{2}\right.\right.}{\lceil 5} = \frac{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\left\lceil\frac{1}{2}\left\lceil\frac{1}{2}\right.\right.}{5.4.3.2.1}$$

$$\beta\left(\frac{9}{2},\frac{1}{2}\right) = \frac{105\pi}{384}$$

b) In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t at t = 0 and i = 0 and L, R and E are constants. **[6]**

ANS: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

∴ Its solution is $ie^{\int R/L\, dt} = \int e^{\int R/L\, dt}.\frac{E}{L}.dt + c$

$$i.e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\, dt + c = \frac{E}{L}.e^{Rt/L}\frac{L}{R} + c$$

$$= \frac{E}{R}e^{Rt/L} + c$$

When t = 0 and i = 0 $\therefore c = -\frac{E}{R}$

$$\therefore i.e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R}$$

$$\therefore i = \frac{E}{R}(e^{Rt/L} - 1)$$

$$\therefore \boldsymbol{i = \frac{E}{R}(1 - e^{-Rt/L})}$$

c) Evaluate $\int_0^6 \frac{dx}{1+3x}$ by using 1} Trapezoidal 2} Simpsons (1/3) rd. and 3} Simpsons (3/8) Th rule. [8]

ANS:

X	0	1	2	3	4	5	6
Y	1	0.25	0.1428	0.1	0.0769	0.0625	0.0526
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1} Trapezoidal Rule:

I = $\frac{h}{2}(X + 2R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

R = Sum of Remaining values = 0.25 + 0.1428 + 0.1 + 0.0769 + 0.0625 = 0.6322

I = $\frac{1}{2}\left(1.0526 + 2(0.6322)\right)$

$\boldsymbol{I = 1.1585}$

2} Simpsons (1/3) rd rule

$I = \frac{h}{3}(X + 2E + 4O)$

X = Sum of Extreme values =1 + 0.0526 = 1.0526

E = Sum of even ordinates = 0.1428 + 0.0769 = 0.2197

O = Sum of odd ordinates = 0.25 + 0.1 + 0.0625 = 0.4125

$$I = \frac{1}{3}\left(1.0526 + 2(0.2197) + 4(0.4125)\right)$$

$\boldsymbol{I = 0.5616}$.

3} Simpsons (3/8) Th rule.

$I = \frac{3h}{8}(X + 2T + 4R)$

X = Sum of extreme value = 1 + 0.0526 = 1.0526

T = Sum of multiple of three =0.1

R = Sum of Remaining values =0.25 + 0.1428 +0.0769 + 0.0625 = 0.5322

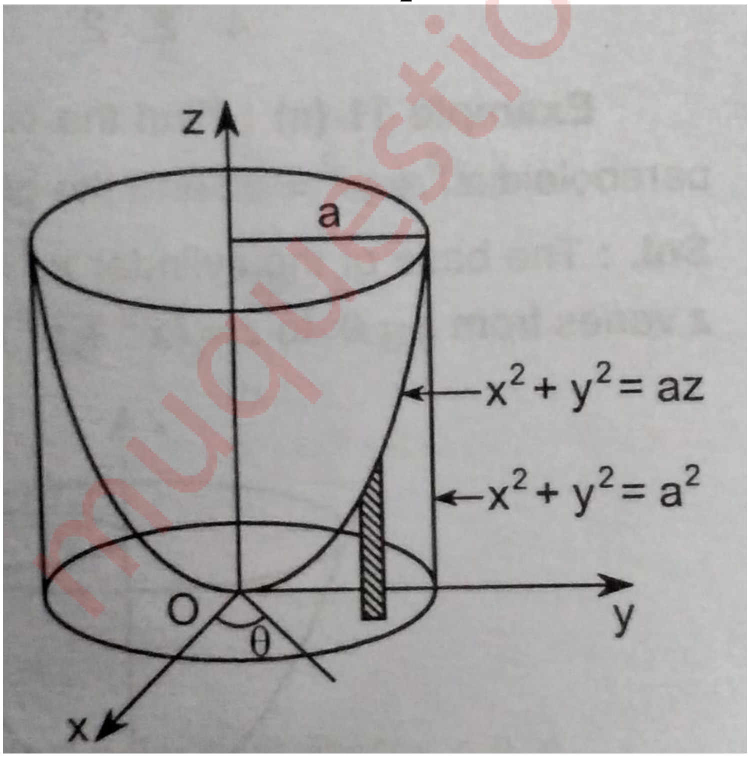
$I = \frac{3*1}{8}(1.0526 + 2(0.1) + 4(0.5322))$

$\boldsymbol{I = 1.06845}$.

Q.6 a) Find the volume bounded by the paraboloid $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$. [6]

ANS: The equations of the cylinder and the paraboloid in polar form are r = a and r^2 = az.

Now, z varies from z = 0 to z = r^2/a, r varies from r = 0 to r = a and θ varies from θ = 0 to θ = $\frac{\pi}{2}$ taken 4 times.



$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\int_{z=0}^{r^2/a} r\,dr\,d\theta\,dz$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}} r\,[z]_0^{r^2/a} dr d\theta$$

$$\therefore V = 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^3}{a}dr\,d\theta$$

$$\therefore V = \frac{4}{a}\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$\therefore V = \frac{4}{a}\int_0^{\frac{\pi}{2}}\frac{a^4}{4}\ d\theta$$

$$\therefore V = a^3\int_0^{\frac{\pi}{2}} d\theta$$

$$\therefore V = a^3[\theta]_0^{\pi/2}$$

$$\boldsymbol{\therefore V = \frac{\pi a^3}{2}}$$

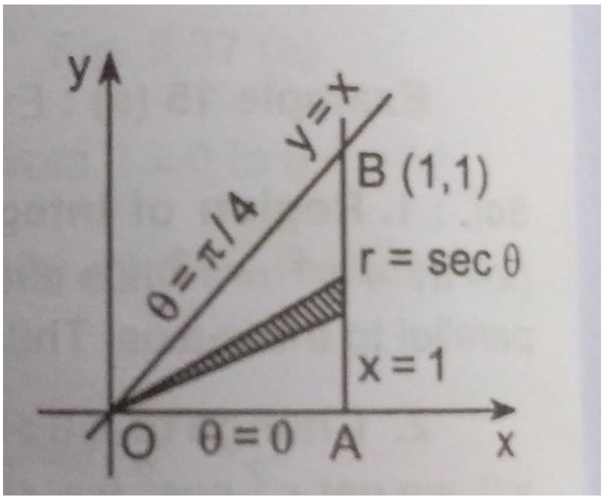
b) Change to polar coordinates and evaluate $\int_0^1\int_0^x (x+y)dydx$. **[6]**

ANS: 1) Region of integration: y= 0 is the x-axis and y = x is a line OB through the origin; x = 0 is the y-axis and x = 1 is a line AB parallel to the y-axis. Thus the region of integration is the triangle OAB.

2) Change to r, θ: Putting x = r cos θ and y = r sin θ, the line y = x becomes r sin θ= r cos θ i.e. tan θ = 1 i.e. θ = $\frac{\pi}{4}$. The x-axis is given by θ = 0 and the y-axis is given by θ =$\frac{\pi}{2}$. And line x = 1 is given by r cos θ = 1 i.e. r = sec θ.

3) Integrand: Putting x = r cos θ and y = r sin θ in (x + y), we get,

r cos θ + r sin θ = r(cos θ + sin θ) and dydx is replaced by r drdθ



$$\therefore I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \mathrm{r}(\cos\theta + \sin\theta)\mathrm{r}dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (\cos\theta + \sin\theta)\mathrm{r}^2 dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta$$

$$I = \frac{1}{3}\int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\, sec^3\theta\, d\theta$$

$$I = \frac{1}{3}[\int_0^{\frac{\pi}{4}} sec^2\theta\, d\theta + \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta} \sin\theta\, d\theta]$$

$$I = \frac{1}{3}\left[\tan\theta + \frac{1}{2cos^2\theta}\right]_0^{\frac{\pi}{4}}$$

$$I = \frac{1}{3}(1 + \frac{1}{2}(2\text{-}1))$$

$$\boldsymbol{I = \frac{1}{2}}$$

c) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$$ **[8]**

ANS: A.E: $D^2 + 3D + 2 = 0$

Solving the equation, we get

$\therefore$ D = -1, -2.

$\therefore$ C.F = $C_1 e^{-x} + C_2 e^{-2x}$.

$\therefore y_1 = e^{-x} \quad y_2 = e^{-2x}$

$\therefore y_1' = -e^{-x} \qquad y_2' = -2e^{-2x}$

$$\therefore w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x}$$

X $= e^{e^x}$.

$$\therefore u = -\int \frac{y_2 X}{w}\, dx$$

$$= -\int \frac{e^{-2x} e^{e^x}}{-e^{-3x}}\, dx$$

$$= -\int e^{e^x} . e^x\, dx$$

Put e^x = t

e^x dx =dt

$\therefore \int e^t dt = e^t + c$.

$\therefore w = e^{e^x} + c$

$$v = \int \frac{y_1 X}{w}\, dx$$

$$v = \int \frac{e^{-x} e^{e^x}}{-e^{-3x}}\, dx$$

$$v = \int e^{e^x} e^{2x}\, dx$$

Putting e^x = t

$$\therefore v = \int e^t . t\, dt = te^t - e^t$$

$$\therefore v = e^x e^{e^x} - e^{e^x}$$

$\therefore$ P.I. = $uy_1 + vy_2 = e^{e^x}.e^{-x} - (e^x e^{e^x} - e^{e^x})\, e^{-2x}$

$= e^{-2x}.\, e^{e^x}$

$\therefore$ The complete solution is,

y = C.F + P.I

$y = C_1e^{-x} + C_2e^{-2x} + e^{-2x}.\, e^{e^x}$