MUMBAI UNIVERSITY CBCGS

APPLIED MATHEMATICS I MAY 2019 PAPER SOLUTIONS

Q1)a) If $\mathrm{u} = \log\left(\frac{x}{y}\right) + \log\left(\frac{y}{x}\right)$ **, find** $\frac{\partial u / \partial x}{\partial u / \partial y}$ **.** **(3M)**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u = 0$$

Ans :

$$\therefore \frac{\partial u / \partial x}{\partial u / \partial y} = \frac{-y}{x}$$

Q1)b) Find the value of $\tanh(\log x)$ **if** $x = \sqrt{3}$ **.** **(3M)**

Ans : Let

$$z = \tanh(\log\sqrt{3})$$
$$\therefore \tanh^{-1} z = \log\sqrt{3}$$
$$\therefore \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\log 3$$
$$\therefore \log\left(\frac{1+z}{1-z}\right) = \log\sqrt{3}$$

By componendo and dividendo

$$\frac{2}{-2z} = \frac{3+1}{1-3}$$
$$\therefore z = \frac{3-1}{3+1}$$

$$\therefore \tanh\log\sqrt{3} = \frac{3-1}{3+1} = \frac{1}{2} \quad .$$

Q1)c) Evaluate $\lim_{x\to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right]$ **.** **(3M)**

Ans: $\lim_{x\to 3}\left[\frac{1}{x-3} - \frac{1}{\log(x-2)}\right]\left[\infty - \infty\right] = \lim_{x\to 3}\frac{\log(x-2)-(x-3)}{(x-3)\log(x-2)}\frac{0}{0}$

$$= \lim_{x\to 3} \frac{\frac{1}{x-2}-1}{\log(x-2)+\frac{(x-3)}{(x-2)}} = \lim_{x\to 3} \frac{-x+3}{(x-2)\log(x-2)+(x-3)} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 3} \frac{-1}{\frac{(x-2)}{(x-2)}+\log(x-2)+1} = -\frac{1}{2} \ .$$

Q1)d) If $u = r^2 \cos 2\theta, v = r^2 \sin 2\theta$ **, find** $\frac{\partial(u,v)}{\partial(r,\theta)}$ **.** **(3M)**

Ans: We have $\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} . \frac{\partial(u,v)}{\partial(r,\theta)}$. But $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$

And $\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2r\cos 2\theta & -2r^2 \sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix} = 4r^3$.

Q1)e) Express the matrix A = $\begin{pmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{pmatrix}$ **as the sum of a Hermitian and a Skew-Hermitian matrix.** **(4M)**

Ans: We have

$$A' = \begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix}$$

$$\therefore A^{\theta} = (\bar{A}') = \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$$

$$\therefore A + A^{\theta} = \begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}$$

$$A - A^{\theta} = \begin{bmatrix} 6i & 2-2i & -4+3i \\ -2-2i & 0 & -1+7i \\ 4+3i & 1+7i & -2i \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^{\theta}) . Q = \frac{1}{2}(A - A^{\theta})$.

But, we know that P is Hermitian and Q is Skew-Hermitian and A = P + Q .

$$\therefore A = P + Q = \begin{bmatrix} 2 & 1+i & (4+3i)/2 \\ 1-i & 0 & (3-3i)/2 \\ (4-3i)/2 & (3+3i)/2 & 0 \end{bmatrix} + \begin{bmatrix} 3i & 1-i & (-4+3i)/2 \\ -1-i & 0 & (-1+7i)/2 \\ (4+3i)/2 & (1+7i)/2 & -i \end{bmatrix}.$$

Q1)f) Expand $\tan^{-1} x$ in powers of $\left(x - \frac{\pi}{4}\right)$. **(4M)**

Ans: Let

$$f(x) = \tan^{-1} x, \text{a} = \frac{\pi}{4}$$

$$\therefore f(x) = \tan^{-1} x,\ f'(x) = \frac{1}{1+x^2},\ f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$\therefore f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right), f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\pi/4\right)^2}, f''\left(\frac{\pi}{4}\right) = -\frac{\pi/2}{\left[1+\left(\pi^2/16\right)\right]^2}, etc$$

$$f(x) = \text{f(a)} + (\text{x}-\text{a})\,\text{f'(a)} + \frac{(x-a)^2}{2!} f''(a) + ...$$

$$\therefore \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right).\frac{1}{1+\left(\pi/4\right)^2} - \left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^2.\frac{1}{\left[1+\left(\pi^2/16\right)\right]^2} + ...$$

Q2)a) Expand $\sin^7\theta$ in a series of sines of multiples of θ . **(6M)**

$$x=\cos\theta+i\sin\theta$$

$$\therefore \frac{1}{x}=\cos\theta-i\sin\theta$$

$$x+\frac{1}{x}=2\cos\theta$$

$$x-\frac{1}{x}=2i\sin\theta$$

$$x^n=\cos n\theta+i\sin n\theta$$

$$\frac{1}{x^n}=\cos n\theta-i\sin n\theta$$

$$\therefore x^n+\frac{1}{x^n}=2\cos n\theta$$

$$x^n-\frac{1}{x^n}=2i\sin n\theta$$

Now, by Binomial Theorem

$$(2i\sin\theta)^7=\left(x-\frac{1}{x}\right)^7=x^7-7x^6.\frac{1}{x}+21x^5.\frac{1}{x^2}-35x^4.\frac{1}{x^4}-21x^2.\frac{1}{x^5}+7x.\frac{1}{x^6}-\frac{1}{x^7}$$

$$\therefore (2i\sin\theta)^7=x^7-7x^5+21x^3-35x+\frac{35}{x}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}$$

$$\therefore (2i\sin\theta)^7=\left(x^7-\frac{1}{x^7}\right)-7\left(x^5-\frac{1}{x^5}\right)+21\left(x^3-\frac{1}{x^3}\right)-35\left(x-\frac{1}{x}\right)$$

$$\therefore (2i\sin\theta)^7=2i\sin7\theta-7.(2i\sin5\theta)+21.\left(2i\sin3\theta\right)-35.\left(2i\sin\theta\right)$$

$$\therefore -2^6\sin^7\theta=\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta$$

$$\therefore \sin^7\theta=\frac{-1}{2^6}\left(\sin7\theta-7\sin5\theta+21\sin3\theta-35\sin\theta\right)$$

Q2)b) If $y=\sin^2x\cos^3x$, then find y_n . **(6M)**

Ans: We have

$$y=\sin^2x\cos^3x$$

$$\therefore y=\sin^2x\cos^2x.\cos x=\frac{1}{4}(\sin2x)^2\cos x$$

$$\therefore y=\frac{1}{8}(1-\cos4x)\cos x=\frac{1}{8}(\cos x-\cos4x\cos x)$$

$$\therefore y=\frac{1}{8}\cos x-\frac{1}{16}\left(\cos5x+\cos3x\right)$$

By using the result $y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16}.5^n \cos\left(5x + \frac{n\pi}{2}\right) - \frac{1}{16}.3^n \cos\left(3x + \frac{n\pi}{2}\right)$$

Q2)c) Find the stationary values of $x^3 + y^3 - 3axy, a > 0$. **(8M)**

Ans: We have $f(x,y) = x^3 + y^3 - 3axy.$

Step 1:

$$f_x = 3x^2 - 3ay,\ f_y = 3y^2 - 3ax$$
$$r = f_{xx} = 6x, s = f_{xy} = -3a, t = f_{yy} = 6y$$

Step 2: We now solve,

$$f_x = 0,\ f_y = 0$$
$$\therefore x^2 - ay = 0, y^2 - ax = 0$$

To eliminate y, we put $y = \frac{x^2}{a}$ in the second equation .

$$\therefore x^4 - a^3x = 0,$$
$$\therefore x(x^3 - a^3) = 0$$

Hence, x=0 or x=a

When x=0,y=0 and when x=a, y=a.

Hence, (0,0) and (a,a) are stationary points.

Step 3: (i) For x=0, y=0, $r = f_{xx} = 0, s = f_{xy} = -3a, t = f_{yy} = 0$.

Hence, $rt - s^2 = 0 - 9a < 0$.

Hence, f(x,y) is neither maximum nor minimum. It is a saddle point.

(ii) For x=a,y=a,

$$r = f_{xx} = 6a, s = f_{xy} = -3a, t = f_{yy} = 6a$$

$\therefore rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$ Hence, f(x,y) is stationary at x=a, y=a .

And $r = f_{xx} = 6a > 0, \because \text{a} > 0$

Hence f(x,y) is minimum at x=a, y=a .

Putting x=a, y=a in x^3+y^3-3axy, the minimum value of

$$f(x,y)=a^3+a^3-3a^3=-a^3 \quad .$$

Q3)a) Compute the real root of $x\log_{10}^{x}-1.2=0$ correct to three places of decimals using Newton-Raphson method. **(6M)**

Ans: We first note that $f(x)=x\log_{10}^{x}-1.2$.

$$\therefore f(1)=1\log_{10}^{1}-1.2=-1.2,\ f(2)=2\log_{10}^{2}-1.2=-0.5979$$
$$f(3)=3\log_{10}^{3}-1.2=0.2313$$

Since f(x) changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

Now, $f'(x)=x.\dfrac{1}{x\log_{e}^{10}}+\log_{10}^{x}=(\log_{e}^{10})^{-1}+\log_{10}^{x}=0.4343+\log_{10}^{x}$

Hence, by Newton-Raphson formula,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},n=0,1,2,3,.....$$

$$=x_n-\frac{x\log_{10}^{x}-1.2}{0.4343+\log_{10}^{x}}$$

For $x_0 = 3$, $x_1=3-\dfrac{3\log_{10}^{3}-1.2}{0.4343+\log_{10}^{3}}=2.74615$

For $x_1 = 2.74615$, $x_2=2.74615-\dfrac{(2.74615).\log(2.74615)-1.2}{0.4343+\log 2.74615}=2.7406$.

For $x_2=2.7406$, $x_3=2.7406$ Hence x = 2.7406 .

Q3)b) Show that the system of equations

$2x-2y+z=\lambda x, 2x-3y+2z=\lambda y, -x+2y=\lambda z$ **can possess a non-trivial solution only if $\lambda=1,\lambda=-3$. Obtain the general solution in each case.** **(6M)**

Ans: We have $\begin{pmatrix}2-\lambda & -2 & 1\\ 2 & -3-\lambda & 2\\ -1 & 2 & -\lambda\end{pmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

The system has non-trivial solution if the rank of A is less than the number of unknowns.

The rank of A will be less than three if |A|=0.

Now, $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\therefore (2-\lambda)(\lambda^2+3\lambda-4)+2(-2\lambda+2)+1(4-3-\lambda)=0$

$\therefore (2-\lambda)(\lambda+4)(\lambda-1)-4(\lambda-1)-(\lambda-1)=0$

$\therefore (\lambda-1)[2\lambda+8-\lambda^2-4\lambda-4-1]=0$

$\therefore (\lambda-1)(-\lambda^2-2\lambda+3)=0$

$\therefore (\lambda-1)(\lambda-1)(\lambda+3)=0$

$\therefore \lambda=1, \lambda=-3$

(i) If λ=1, we have,

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By $R_2 \rightarrow R_2-2R_1$, $R_3 \rightarrow R_3+R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴x-2y+z=0. Putting $z=t_1$, $y=t_2$.

∴The solution is $x=2t_2-t_1, y=t_2, z=t_1$.

Q3)c) If $\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$ **, prove that** $\alpha=\frac{n\pi}{2}+\frac{\pi}{4}$ **and** $\beta=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ **.**

(8M)

Ans: We have

$$\tan(\alpha+i\beta)=\cos\theta+i\sin\theta$$
$$\therefore \tan(\alpha-i\beta)=\cos\theta-i\sin\theta$$
$$\therefore \tan 2\alpha=\tan\left[(\alpha+i\beta)+(\alpha-i\beta)\right]$$
$$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\tan(\alpha-i\beta)}=\frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}$$

$$\therefore \tan 2\alpha = \frac{2\cos\theta}{0}$$

$$\therefore 2\alpha = \frac{\pi}{2}$$

$$2\alpha = n\pi + \frac{\pi}{2}$$

$$\therefore \alpha = \frac{n\pi}{2} + \frac{\pi}{4}$$

$$\tan(2i\beta) = \tan\left[(\alpha + i\beta) - (\alpha - i\beta)\right]$$

$$= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta).\tan(\alpha - i\beta)} = \frac{2i\sin\theta}{1+1} = i\sin\theta$$

$$\therefore i\tan(\mathrm{h}\,2\beta) = i\sin\theta$$

$$\therefore \tan(h2\beta) = \sin\theta$$

$$\therefore 2\beta = \tanh^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$$

But

$$1+\sin\theta = \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2$$

$$1-\sin\theta = \left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2$$

$$\therefore 2\beta = \frac{1}{2}\log\left[\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right]^2 = \log\left[\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right]$$

$$\therefore \beta = \frac{1}{2}\log\left[\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right] = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Q4)a) Using the encoding matrix as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, encode and decode the message MOVE . **(6M)**

Ans: Step 1: To replace letters by numbers

M	O	V	E
13	15	22	5

We write this in a sequence of 2 X 2 matrix

$$\begin{bmatrix}13\\15\end{bmatrix}\begin{bmatrix}22\\5\end{bmatrix}.$$

Step 2 : To encode the message

We now premultiply each of the above column-vectors by encoding matrix $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$.

$$\therefore\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}13 & 22\\15 & 5\end{bmatrix}=\begin{bmatrix}13+15 & 22+5\\0+15 & 0+5\end{bmatrix}=\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}.$$

The above message is transmitted in the following linear form taking numbers column-wise. The message is transmitted in the linear form as

28 15 27 5

Step 3: To decode the message :

The above received message is now written in a sequence of 2 X 1 column matrix as

$$\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}.$$

The above matrix is then premultiplied by the inverse of the coding matrix i.e., by

$$\therefore\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}28 & 27\\15 & 5\end{bmatrix}=\begin{bmatrix}28-15 & 27-5\\0+15 & 0+5\end{bmatrix}=\begin{bmatrix}13 & 22\\15 & 5\end{bmatrix}.$$

Step 4 : To replace numbers by letters

The columns of this matrix are written in linear form as

14 15 23 27 19 20 21 4 25 27

Now it is transformed into letters using corresponding alphabets

14	15	23	27	19	20	21	4	25	27
N	O	W	*	S	T	U	D	Y	*

This is the required message.

Q4)b) If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$ **then prove that** $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ **.** **(6M)**

Ans: Let $X = e^{x-y}, Y = e^{y-z}, Z = e^{z-x}$. Then u = f(X,Y,Z)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z} e^{z-x}(-1)$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Z} e^{z-x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} e^{x-y}(-1) + \frac{\partial u}{\partial Y} e^{y-z}(1) + \frac{\partial u}{\partial Z}(0)$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Y} e^{y-z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y} \mathrm{e}^{y-z}(-1) + \frac{\partial u}{\partial Z} e^{z-x}(1)$$

$$\therefore \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} e^{y-z} - \frac{\partial u}{\partial Z} e^{z-x}$$

Q4)c) If $y = a\cos(\log x) + b\sin(\log x)$ **, then show that** $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

(8M)

Ans: We have

$$y = a\cos(\log x) + b\sin(\log x)$$

$$\therefore y_1 = -a\sin(\log x).\frac{1}{x} + b\cos(\log x).\frac{1}{x}$$

$$\therefore xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t x ,

$$\therefore xy_2 + y_1 = -a\cos(\log x).\frac{1}{x} - b\sin(\log x).\frac{1}{x}$$

$$\therefore x^2 y_2 + xy_1 + y = 0 \quad .$$

Applying Leibnitz's theorem to each term, we get

$$x^2 y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + [xy_{n+1} + n(1)y_n] + y_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 - n + n + 1)\,\mathrm{y}_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + 1)y_n = 0$$

Q5)a) If 1, $\alpha, \alpha^2, \alpha^3, \alpha^4$ are the roots of $x^5 - 1 = 0$, find them and show that

(1-α)(1-α^2)(1-α^3)(1-α^4)=5. **(6M)**

Ans: We have

$$x^5 = 1 = \cos 0 + i\sin 0$$

$$\therefore x^5 = \cos(2k\pi + i\sin(2k\pi)$$

$$\therefore x = (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}} = \cos\frac{2k\pi}{5} + \sin\frac{2k\pi}{5}$$

Putting k = 0, 1, 2, 3, 4, we get the five roots as

$x_0 = \cos 0 + i\sin 0$

$$\therefore x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ = we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$.

Therefore, the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore \frac{x^5 - 1}{x - 1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \mathrm{x}^4 + x^3 + x^2 + x + 1$$

Putting x = 1, we get

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

Q5)b) If $\theta = t^n e^{\frac{-r^2}{4t}}$ **,**

Find n which will make $\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$ **.** **(6M)**

Ans:

$$\frac{\partial\theta}{\partial t} = nt^{n-1}.e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}}.\left(\frac{-r^2}{4}\right)\left(\frac{-1}{t^2}\right)$$

$$= \frac{n}{t}.t^n.\frac{\theta}{t^n} + t^n.\frac{\theta}{t^n}\left(\frac{r^2}{4t^2}\right)$$

$$= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta$$

$$\left(\because e^{\frac{-r^2}{4t}} = \frac{\theta}{t^n}\right)$$

Also ,

$$\frac{\partial\theta}{\partial r} = t^n.e^{\frac{-r^2}{4t}}.\left(\frac{-2r}{4t}\right) = \frac{-r\theta}{2t} \quad .$$

$$\therefore r^2\frac{\partial r}{\partial\theta} = \frac{-r^3\theta}{2t}$$

$$\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{-r^3\theta}{2t}\right) = \frac{-1}{2t}.\frac{\partial}{\partial r}(r^3\theta) = \frac{-1}{2t}\left[r^3\frac{\partial\theta}{\partial r} + 3r^2\theta\right]$$

$$= \frac{-1}{2t}\left[\frac{-r^4\theta}{2t} + 3r^2\theta\right] = r^2\left(\frac{r^2}{4t^2} - \frac{3}{2t}\right)\theta$$

$$\therefore \frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \left(\frac{r^2}{4t^2} - \frac{3}{2t}\right)\theta$$

Now equating we get

$$\frac{n}{t} = \frac{-3}{2t}$$

$$\therefore n = \frac{-3}{2}$$

Q5)c) Find the root (correct to three places of decimals) of $x^3 - 4x + 9 = 0$ lying between 2 and 3 by using Regula-Falsi method . (8M)

Ans: Let $y = f(x) = x^3 - 4x - 9$. Here, $x_1 = 2$ and $x_2 = 3$.

$$\therefore y_1 = f(x_1) = f(2) = 2^3 - 4(2) - 9 = 8 - 8 - 9 = -9 < 0$$
$$y_2 = f(x_2) = f(3) = 3^3 - 4(3) - 9 = 27 - 12 - 9 = 6 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2 to 3 , there is a root between 2 and 3 .

The root is given by

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(6) - 3(-9)}{6 - (-9)} = \frac{39}{15} = 2.6 \ .$$

Now, $y_3 = f(x_3) = f(2.6) = (2.6)^3 - 4(2.6) - 9 = -1.82 > 0$

Since f(x) changes its sign from negative to positive as x goes from 2.6 to 3 , there is a root between 2.6 and 3 .

First Iteration : Let

$$x_1 = 2.6, x_2 = 3, y_1 = -1.82, y_2 = 6$$
$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.6(6) - 3(-1.82)}{6 - (-1.82)} = 2.693$$
$$y_3 = f(x_3) = (2.693)^3 - 4(2.693) - 9 = -0.242 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2.693 to 3 , there is a root between 2.693 to 3 .

Second Iteration : Let

$$x_1 = 2.693, x_2 = 3, y_1 = -0.262, y_2 = 6$$
$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.693(6) - 3(-0.262)}{6 - (-0.262)} = 2.7058 = 2.706$$
$$y_3 = f(x_3) = (2.706)^3 - 4(2.706) - 9 = -0.009 > 0$$

Since f(x) changes its sign from negative to positive as x goes from 2.706 to 3 , there is a root between 2.706 to 3 .

Third Iteration : Let

$$x_1 = 3, x_2 = 2.706, y_1 = 6, y_2 = -0.009$$

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(0.03) - 2.706(6)}{-0.009 - 6} = 2.706$$

Hence, the root correct to three places of decimals = 2.706 .

Q6)a) Find non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **is reduced to normal form. Also find its rank.** **(6M)**

Ans: We first write A = I_3 A I_4

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

By $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ \quad .$$

Hence , the rank of A is 2 .

Q6)b) Find the principle value of $(1+i)^{1-i}$ **.** **(6M)**

Ans: Let

$$z = (1+i)^{1-i}, \therefore \log z = (1-i)\log(1+i)$$

$$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + i\tan^{-1}1\right]$$

$$= (1-i)\left[\frac{1}{2}\log 2 + i.\frac{\pi}{4}\right] = \frac{1}{2}\log 2 + \frac{i\pi}{4} - \frac{i}{2}\log 2 + \frac{\pi}{4}$$

$$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = x + iy$$

$$\therefore z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$$

$$= e^{\left(\frac{1}{2}\right)\log 2 + \left(\frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

$$= \sqrt{2}e^{\frac{\pi}{4}}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

Q6)C) Solve the following equations by Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

(Take three iterations) **(8M)**

Ans: We first write the three equations as

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

(i) First Iteration : We start with the approximation y=0, z=0 and we get

$$\therefore x_1 = \frac{85}{27} = 3.15 \ .$$

We use this approximation to find y_1 i.e. we put x = 3.15 , z=0 in the second equation

$$\therefore y_1 = \frac{1}{15}[72 - 6(3.15)] = 3.54 \ .$$

We use these values of x_1 and y_1 to find z_1 i.e. we put x=3.15 and y=3.54 in the third equation

$$\therefore z_1 = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91 \ .$$

(ii) Second Iteration : We use the latest values of y and z to find x , i.e. we put y= 3.54 , z=1.91 in equation 1 , we get

$$x_2 = \frac{1}{27}[85 - 6(3.54) + 1.91] = 2.43$$

We put x = 2.43 , z = 1.91 to find y from equation 2. Thus ,

$$y_2 = \frac{1}{15}[72 - 6(2.43) - 2(1.91)] = 3.57$$

We put x = 2.43 , y = 3.57 in equation 3 to find z. Thus ,

$$z_2 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93$$

(iii) Third iteration: Putting y = 3.57 , z = 1.93 in equation (1) we get

$$x_3 = \frac{1}{27}[85 - 6(3.57) + 1.93] = 2.43$$

Putting x = 2.43 , z = 1.93 in equation 2 we get

$$y_3 = \frac{1}{15}[72 - 6(2.43) - 2(1.93)] = 3.57$$

Putting x=2.43 , y=3.57 in equation 3 we get

$$z_3 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93 \ .$$

Since the second and third iteration give the same values

x = 2.43 , y = 3.57 , z = 1.93
