Q.P CODE: 37525

APPLIED MATHEMATICS - IV SOLUTION

(CBCGS SEM -4 MAY 2019)

BRANCH – ELECTRONICS &TELECOMMUNICATION

Q1) If X_1 has a a mean 4 and variance 9 and X_2 has a mean -2 variance 4 ,

And two are independent , Find $E(2X_1 + X_2 - 3)$ and $V(2X_1 + X_2 - 3$

Solution: **[5]**

We have $E(X_1) = 4, V(X_1) = 9$, $E(X_2) = -2$ and $V(X_2) = 4$

$$\therefore E(2X_1 + X_2 - 3) = E(2X_1 + X_2) - 3 = 2E(X_1) + E(X_2) - 3$$

$$= 2(4) + (-2) - 3 = 3$$

$$V(2X_1 + X_2 - 3) = V(2X_1 + X_2)$$

$$= 2^2 V(X_1) + V(X_2)$$

$$= 4(9) + 5 = 41$$

(b) Find the extremal of $\int_{x_1}^{x_2} (x + y')y' dx$ **[5]**

Solution

We have $F = xy' + y'^2$

Since F does 'not contain y explicity

Now , $F = xy' + y'^2 \therefore \frac{\partial F}{\partial y} = x + 2y'$

But from the formula $\frac{\partial F}{\partial y'} = c \quad \therefore x + 2y' = c$

$\therefore 2\frac{dy}{dx} = c - x \qquad \therefore \frac{dy}{dx} = \frac{c}{2} - \frac{x}{2} \qquad \therefore dy = \left(\frac{c}{2} - \frac{x}{2}\right) dx$

By the integration, we get $y = \frac{c}{2}x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c_2$

Taking the arbitrary constant suitably , $\boxed{y = -\frac{x^2}{4} + c_1 x + c_2}$

(c) Verify Cauchy Schwartz inequality for the vectors = (−4, 2, 1) and

***w* = (8, −4, −2)** **[5]**

<u>Solution</u>

We have $\|u\| = \sqrt{16+4+1} = \sqrt{21}$ and $\|v\| = \sqrt{64+16+4} = \sqrt{84}$

$\therefore \|u\|\,\|v\| = \sqrt{21}\,\sqrt{84} = 42$

And $|u\,.\,v| = |u_1 v_1 + u_2 v_2 + u_3 v_3|$

$= |(-4)(8) + (2)(-4) + (1)(-2)|$

$= |-32 - 8 - 2| = 42$

$\therefore \|u\|\|v\| = |u\,.\,v|$

By Cauchy – Schwartz inequality we should have $|u\,.\,v| \leq \|u\|\|v\|$.

Hence, Cauchy – Schwartz inequality holds good for the given vectors.

(d) **Check whether A =** $\begin{bmatrix} \mathbf{2} & \mathbf{-2} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ **is derogatory or not .** **[5]**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\therefore \quad (2-\lambda)[(-(1-\lambda)(1+\lambda)-3]+2[(-(1+\lambda)-1]+3[3-(1-\lambda)]=0$

$(2-\lambda)[-4+\lambda^2]-2[-2+2\lambda]+3[2+3\lambda]=0$

$\lambda^3 — 2\lambda^2-5\lambda+6=0 \quad \therefore \lambda^3-\lambda^2-\lambda^2+\lambda-6\lambda+6=0$

$(\lambda-1)(\lambda^2-\lambda-6)=0 \quad \therefore\ (\lambda-1)(\lambda-3)(\lambda-2)=0$

$\therefore \lambda = 1, -2, 3$

Since all the roots are distinct and since the characteristic equation are distinct. $f(x)=(x-1)(x+2)(x-3)$ is minimal polynomial . The degree of minimal equaltion is equal to 3. Hence the matrix is non-derogatory

Q2)

(a) **Using Cauchy's Residue theorem evaluate** $\int_c \frac{z-1}{(z+1)^2(z-2)}$ **where C is** $|\mathbf{z}| = \mathbf{4}$ **[6]**

Solution:

Clearly $z=-1$ is a pole of order 2 and $z=2$ is a simple pole.

Residue at $(z = 2)$ = $\lim_{z\to 2}\left[\frac{(z-2)(z-1)}{(z+1)^2(z-2)}\right] = \lim_{z\to 2}\frac{z-1}{(z+1^2)} = \frac{1}{9}$

Residue (at $z = -1$) $= \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z-1}{(z+1)^2(z-2)}\right]$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z-1}{z-2}\right) = \lim_{z\to -1}\left[\frac{(z-2)\cdot 1-(z-1).1}{(z-2)^2}\right] = \lim_{z\to -1} -\frac{1}{9}$$

$\therefore \int_c f(z)dz = 2\pi i$(Sum of the residues)

$$= 2\pi i\left(\frac{1}{9}-\frac{1}{9}\right) = 0$$

(b) **Show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2}(y')^2\, dx$ subject to the condition $\int_{x_1}^{x_2} y\, dx = k$ is a parabola .** **[6]**

Solution :

We have to find $y = f(x)$such that

$\int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} y'^2\, dx$……………………………………………(1)

is minimum subject to the condition $\int_{x_1}^{x_2} G\, dx = \int_{x_1}^{x_2} y\, dx = k$…………..(2)

To use Lagrange's equation , we multiply (2) by λ and add it to (1)

$\therefore H = F + \lambda G = \int_{x_1}^{x_2}(y'^2 + \lambda y)\, dx$…………………………………………… (3)

Since the integrand is free from x , we use

$F - y'\frac{\partial F}{\partial y'} = c$……………………………………………(4)

Where, $F = H = y'^2 + \lambda y$ ……………………………………………(5)

Hence from (4) using (5) we get

$y'^2 + \lambda y - y'.2y' = c \quad \therefore -y'^2 + \lambda y = c$

$\therefore y'^2 - \lambda y = -c = c_1$

$\therefore y' = \sqrt{c_1 + \lambda y} \qquad \therefore \frac{dy}{\sqrt{c_1+\lambda y}} = dx$

Integrating we get

$\frac{2}{\lambda}\sqrt{c_1 + \lambda y} = x + c_2 \qquad \therefore \sqrt{c_1 + \lambda y} = \frac{\lambda}{2}(x + c_2)$

$(c_1 + \lambda y) = \left(\frac{\lambda}{2}\right)^2 (x + c_2)^2 \quad \therefore \lambda y = \frac{\lambda^2}{2}x^2 + \frac{\lambda^2}{2}c_2 x + \frac{\lambda^2}{4}c_2^2 - c_1$

$\therefore y = \frac{\lambda}{4}x^2 + \frac{c_2\lambda}{2}x + \left(\frac{\lambda}{4}c_2^2 - \frac{c_1}{\lambda}\right) = \frac{\lambda}{4}x^2 + c'x + c"$

This is a parabola..

(c) Is the matrix $A = \begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{vmatrix}$ **diagonizable ? If so find the diagonal matrix and the transforming matrix. [8]**

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\therefore (2-\lambda)[(2-\lambda)(1-\lambda) - 0] - 1[1(1-\lambda) - 0] + 1[0-0] = 0$

$\therefore (2-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda) = 0$

$\therefore (1-\lambda)[(2-\lambda)(2-\lambda)-1]=0 \quad \therefore (1-\lambda)(4-4\lambda+\lambda^2-1)=0$

$\therefore (1-\lambda)(\lambda^2-4\lambda+3)=0$

$\therefore (1-\lambda)(\lambda-3)(\lambda-1)=0 \qquad \therefore \lambda = 1,1,3$

Since the Eigen values are repeated the matrix A may be or may not be diagonalisable.

(i) For $\lambda = 3$,$[\,A-\lambda_1 I]\,X=0$ gives

$$\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

by $\quad R_2+R_1 \begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 2\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $\quad R_3+R_2 \,\&\, \frac{1}{2}R_2 \begin{bmatrix}-1 & 1 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore -x_1+x_2+x_3=0 \quad ,x_3=0$

Putting $x_2=t$, we get $x_1=t$

$$X_1=\begin{bmatrix}t\\t\\0\end{bmatrix}=t\begin{bmatrix}1\\1\\0\end{bmatrix} \quad \therefore \text{Eigenvector is } [1,1,0]$$

There are three variables and the rank is 2, hence , there is only $3-2=1$ is the independent solution .

For $\lambda = 3$ alegrbric multiplicity =1 and the geometric multiplicity -=1

(ii) For $\lambda = 1$,$[A - \lambda_2 I]\ X = 0$ gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore$ $x_1 + x_2 + x_3 = 0$

Let $x_2 = -s$, $x_3 = -t$ $\therefore x_1 = s + t$

$$\therefore X_2 = \begin{bmatrix} s+t \\ -s+0 \\ 0-t \end{bmatrix} = \begin{bmatrix} s \\ -s \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ and } \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

There are three variables and the rank of the matrix is one and hence there are $3 - 1 = 2$ independent vectors .

$\therefore$ For λ =1 , since the eigen value is repeated twice , the algebric multiplicity =2 and since X_2 ,$X_3$are two independent vectors corresponding to $\lambda = 1$,the geometry multiplicity $= 2$. Since the algebric multiplicity and geometric multiplicity are equal

then the matrix is diagonalisable .

The diagonalising matrix is $M = [X_1 \quad X_2 \quad X_3] \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Thus, the given matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalised to the diagonal matrix

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By transforming matrix, M = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Now, verifying $M^{-1}AM =$. We shall first obtain M^{-1} by elementary transformations.

For this we write $A = IA \ Where \ \ A = M$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}$$

By $R_2 - R_1$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}$

By $R_1 + R_3$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}$

By $R_2 - R_3$, $R_1 + \frac{1}{2}R_2$, $-\frac{1}{2}R_2 - R_3$

We get $M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix}$

Now , $M^{-1}AM = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

Q3)

(a) Verify Cayley – Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ and hence find A^{-1} . [6]

Solution:

The characteristic Equation is $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)[(1+\lambda)(1+\lambda)-4] - 2[-2(1+\lambda)-12] + 3[2+3(1+\lambda)] = 0$

$\therefore (1-\lambda)(-3+2\lambda+\lambda^2) + 2(14+2\lambda) + 3(5+3\lambda) = 0$

$\therefore (-3+2\lambda+\lambda^2+3\lambda-2\lambda^2-\lambda^3+28+4\lambda+15+19\lambda = 0$

$\lambda^3+\lambda^2-18\lambda-40 = 0$ ……………………………………..….(1)

Cayley –Hamilton Theorem states that this equation is satisfies by A i.e

$$A^3 + A^2 + 18A - 40I = 0$$

Now , $A^2 = \begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix} = \begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}$

$$A^3 = \begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix} = \begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix}$$

It can be seen that

$$A^3 + A^2 - 18A - 40I$$

$$= \begin{bmatrix}44 & 33 & 46\\ 24 & 13 & 74\\ 52 & 14 & 8\end{bmatrix} + \begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix} - \begin{bmatrix}18 & 36 & 54\\ 36 & -18 & 72\\ 54 & 18 & -18\end{bmatrix} - \begin{bmatrix}40 & 0 & 0\\ 0 & 40 & 0\\ 0 & 0 & 40\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus the theorem is verified.

(a) Now multiplying (1) by A^{-1} , we get $A^2 + A - 18I - 40A^{-1} = 0$

$\therefore 40A^{-1} = A^2 + A - 18I$……………………………………..(2)

$$\therefore\ 40A^{-1} = \begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix} - 18\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} =$$

$$\begin{bmatrix}-3 & 5 & 11\\ 14 & -10 & 2\\ 5 & 5 & -5\end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

(b) Check whether the following are subspace of R^3 **[6]**

(I) $W = \{(a, 0, 0)\}\ a \in R$ **(II)** $W = \{(x, y, z) | x = 1, z = 1, y \in R\}$

Solution:

(i) Let $v_1 = (a_1, 0,0)$ and $v_2 = (a_2, 0,0)$ be the two vectors in R^3

Now, $v_1 + v_2 = (a_1 , 0 , 0) + (a_2 ,0 , 0) = (a_1 + a_2, 0 , 0)$

Since $a_1 + a_2 \in R$, $v_1 + v_2$ is in R^3

If k is scalar then $kv_1 = k(a_1, 0 , 0) = (ka_1, 0 , 0)$

Hence, kv_1 is also in R^3

Hence, $W = \{(a , 0 , 0)|\ a \in R\}$ is the suspace of R^3.

(ii) Let $v_1 = (x_1, y_1, z_1)$ where $x_1^2 + y_1^2 + z_1^2 \leq 1$

We consider the second condition of theorem ie if k is any scalar then

$$kv_1 = (kx_1, kx_2, kx_3)$$

But if k >1 then $(kx_1)^2 + (kx_2)^2 + (kx_3)^2 = k^2(x_1^2 + y_2^2 + z_3^2)$ is not ≤ 1

Hence , V is not closed under multiplication and hence not a subspace of R^3.

(c) **Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in Taylor's and Laurent's series indicating regions of convergence.** **[8]**

Solution:

Let $f(z) = \frac{a}{z-1} + \frac{b}{z-3}$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{a(z-2)+b(z-1)}{(z-1)(z-2)}$$

$$\therefore 1 = a(z-2) + b(z-1)$$

Putting $z = 1$ we get $1 = a(1-2)$ i.e $a = -1$

Putting $z = 2$ we get $1 = b(2-1)$ ie $b = 1$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

Hence $f(x)$ is not analytic at $z = 1$ and $z = 2$

$\therefore f(x)$ is analytic when $|z| < 1$, $1 < |z| < 2$, $|z| > 2$

Case 1 :When $|z| < 1$, clearly $|z| < 2$ hence

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} - \frac{1}{2-z}$$

$$= \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{\left[1-\frac{z}{2}\right]} = (1-z)^{-1} - \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$$

$$= 1 + z + \frac{z^2}{2!} + \ldots\ldots\ldots - \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2!\cdot 4} + \cdots\right]$$

Case 2 ; $1 < |z| < 2$,

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{2}\right)}$$

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\ \ldots\ldots\ldots..\right]-\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}\ldots\ldots\ldots.\right]$$

which is in the form of Laurent's series.

Case :3 $\quad |z| > 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\ \left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\ \ldots\ldots\ldots\right]-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ \ldots..\right)$$

which is in the form of laurent's series.

Q4)

(a) Using Rayleigh – Ritz method to solve the boundary value problem .

$$\boldsymbol{I = \int_0^1 2xy + y^2 - (y')^2 dx\ ;\ 0 \le x \le 1 \text{given } y(0) = y(1) = 0.} \quad \textbf{[6]}$$

<u>Solution:</u>

We have to extremis $I = \int_0^1 F(x, y, y')dx$ ……………… (1)

$where\ \ F = 2xy + y^2 - y'^2$ ………………………………… (2)

Now assume the trial solution $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ … … . (3)

By the data $\bar{y}(0) = 0 \therefore c_0 = 0\ ;\ \bar{y}(1) = 0 \quad \therefore 0 = c_1 + c_2 \ \therefore c_2 - c_1$

$\therefore \bar{y}(x) = c_1 x - c_2 x^2 = c_1 x(1-x)$………………………… (4)

$$\bar{y}'(x) = c_1 x - 2c_1 x = c_1(1-2x)$$

Putting these values in $I = \int_0^1 (2xy + y^2 - y'^2)dx$ we get

$$I = \int_0^1 \{2x[c_1 x(1-x)] + c_1^2 x^2 (1-x)^2 - c_1^2 (1-2x)^2\}dx$$

$$= c_1 \int_0^1 \{2(x^2 - x^3) + c_1[x^2 - 2x^3 + x^4 - (1 - 4x + 4x^2)]\}dx$$

$$= c_1 \int_0^1 \{2(x^2 - x^3) + c_1[x^2 - 2x^3 + x^4 - (1 - 4x + 4x^2)]\}dx$$

$$= c_1 \left[2\left(\frac{x^3}{3} - \frac{x^4}{4}\right) + c_1\left(-x + 2x^2 - x^3 - \frac{x^4}{2} + \frac{x^5}{5}\right)\right]_0^1$$

$$= c_1 \left[2\left(\frac{1}{3} - \frac{1}{4}\right) + c_1\left(-1 + 2 - 1 - \frac{1}{2} + \frac{1}{5}\right)\right]$$

$\therefore I = c_1\left(\frac{1}{6} - \frac{3}{10}c_1\right) = \frac{c_1}{6} - \frac{3}{10}c_1^2$

Its stationary values are given by :

$$\frac{dI}{dc_1} = 0 \quad \therefore \frac{1}{6} - \frac{3}{5}c_1 = 0 \quad \therefore c_1 = \frac{1}{6} \cdot \frac{5}{3} = \frac{5}{18}.$$

Hence from (4) the approximate solution is $\bar{y}(x) = \frac{5}{18} x(1-x)$

(b) If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3 \tan A = A\, tan 3$. **[6]**

Solution :

The Characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore -(1+\lambda)(1-\lambda)-8=0 \qquad \therefore \lambda^2-9=0 \qquad \therefore \lambda=3,-3$$

(i) For $\lambda=3, [A-\lambda_1 I]X=0$ gives

$$\begin{bmatrix}-4 & 4\\ 2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad \text{by } R_2+\tfrac{1}{2}R_1 \begin{bmatrix}-4 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\therefore -4x_1+4x_2=0 \qquad \therefore x_1-x_2=0$$

Putting $x_2=t$, we get $x_1=t$ $\qquad \therefore X_1=\begin{bmatrix}t\\t\end{bmatrix}=t\begin{bmatrix}1\\1\end{bmatrix}$

$\therefore$ The Eigen vector is [1,1].

(ii) For $\lambda=-3, [A-\lambda_2 I]X=0$ gives

$$\begin{bmatrix}2 & 4\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \qquad \text{by } \ R_2-R_1\begin{bmatrix}2 & 4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\therefore 2x_1+4x_2=0 \quad ie \ \ x_1+2x_2=0$$

Putting $x_2=-t$ and $x_1=-2x_2=2t$

$$\therefore \ X_2=\begin{bmatrix}2t\\-t\end{bmatrix}=t\begin{bmatrix}2\\-1\end{bmatrix}$$

$\therefore$The Eigen vector is $[2,-1]$

$$\therefore M=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix} \text{ And } |M|=-3$$

$$\therefore \qquad M^{-1}=\frac{adj.M}{|M|}=-\frac{1}{3}\begin{bmatrix}-1 & -2\\ -1 & 1\end{bmatrix}$$

$$\text{Now,} D=\begin{bmatrix}3 & 0\\ 3 & -3\end{bmatrix}$$

$$\because f(A)=\tan A, \qquad f(D)=\begin{bmatrix}\tan 3 & 0\\ 0 & -\tan 3\end{bmatrix}$$

$$\therefore \tan A = M\, f(D) M^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tan 3 & 0 \\ 0 & \tan(-3) \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} \tan 3 & -2\tan 3 \\ \tan 3 & -\tan 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} =$$

$$-\frac{1}{3} \begin{bmatrix} \tan 3 & -4\tan 3 \\ -2\tan 3 & -\tan 3 \end{bmatrix}$$

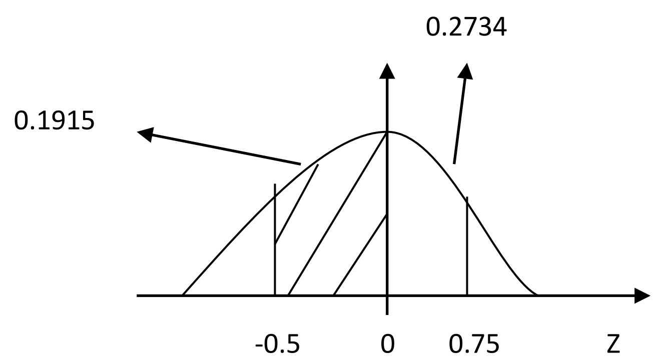
$$\therefore\ 3\tan A = \begin{bmatrix} \tan 3 & 4\tan 3 \\ 2\tan 3 & \tan 3 \end{bmatrix} = \tan 3 \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix} = \tan 3 \,.\, A$$

$$= A \tan 3.$$

(c) If sizes of 10,000 items are normally distributed with mean 20cms & standard deviation of 4cms . Find the probability that an item selected at random will have size:

(1) Between 18 cms and 23 cms (2) above 26 cms [8]

Solution : (1)



We have SNVZ $= \frac{X-M}{\sigma} = \frac{X-20}{4}$

When $X = 18$ $\quad Z = \frac{18-20}{4} = -0.5$

When $X = 23$ $\quad Z = \frac{23-20}{4} = 0.75$

$P(18 \le x \le 23) = P(-0.5 \le z \le 0.75)$_

= area between $(Z = -0.5$ to 0.75

$= 0.1915 + 0.2734 = \ 0.4649$

(2) When $X = 26$

$$Z = \frac{26-20}{4} = \frac{6}{4} = 1.5$$

$$= \text{area to the right of } 1.5$$

$$= 0.5 - 0.432 = 0.0665$$

Q.5)

(a) Find the orthonormal basis of R^3 using Gram-Schmidt process where S={(1,0,0), (3,7,-2), (0,4,1)} **[6]**

Solution :

Step 1: $v_1 = u_1 = (1,0,0)$

Step 2: $v_2 = u_2 - \ proj\ u_2 - \frac{\langle u_2, v_2 \rangle}{\|v_1^2\|} \,.\, v_1$

Now, $\langle u_2, v_1 \rangle = 3 + 0 + 0 = 3$ and $\|v_1\|^2 = 1 + 0 + 0 = 1$

$$v_2 = (3,7,-2) - \frac{3}{1}(1,0,0) = (0,7,-2)$$

Step 3: $v_3 = u_3 - proj\ u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1^2\|} \,.\, v_1 \ \frac{\langle u_3, v_2 \rangle}{\|v_2^2\|} \,.\, v_2$

Now $\langle u_3, v_1 \rangle = 0 + 0 + 0 = 0;$ $\quad \langle u_3, v_2 \rangle = 0 + 28 - 2 = 26$

$$\therefore v_3 = (0,4,1) - 0 - \frac{26}{53}(0,7,-2) = \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

Hence, $v_1 = (1,0,0)$, $v_2 = (0,7,-2)$, $v_3 = \left(0,\frac{30}{53},\frac{105}{53}\right)$ from the orthogonal basis to R^3. Now the norms of this vector are

$\|v_1\| = \sqrt{1+0+0} = 1$; $\|v_2\| = \sqrt{0+49+4} = \sqrt{53}$

$$\|v_3\| = \sqrt{0+\frac{900}{53^2}+\frac{11025}{53^2}} = \frac{15}{\sqrt{53}}$$

And hence the orthogonal basis for R^3 is

$q_1 = \frac{v_1}{\|v_1\|} = (1,0,0)$; $q_2 = \frac{v_2}{\|v_2\|} = \left(0,\frac{0,7}{\sqrt{53}},-\frac{2}{\sqrt{53}}\right)$;

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0,\frac{2}{\sqrt{53}},\frac{7}{\sqrt{53}}\right)$$

(b) In a factory, machines A, B & C produce 30%,50% & 20% of the total production of an item. Out of their production 80%, 50% & 10% are defective respectively. An item is chosen at random and found to be defective respectively. What is the probability that it was produced by machine A.? [6]

Solution:

$$P(A) = \frac{30}{100} = 0.3$$

$$P(B) = \frac{50}{100} = 0.5$$

$$P(C) = \frac{20}{100} = 0.2$$

Defective

$$P'(A) = \frac{80}{100} = 0.8$$

$$P'(B) = \frac{50}{100} = 0.5$$

$$P'(C) = \frac{10}{100} = 0.1$$

From Bays Theorem $= \frac{P(A).P'(A)}{P(A)(P'A)+P(B)P'(B)+P(C)P'(C)}$

=(0.3×0.8)/(0.3×0.8+0.5×0.5+0.2×0.1)

$$= \frac{5}{17} = 0.294$$

(c) Evaluate $\int_{-\infty}^{\infty} \frac{\mathbf{dx}}{(\mathbf{x^2+4})(\mathbf{x^2+9}))}$ **[8]**

Solution:

(i) Consider the contour consisting of a semi circle and diameter on the real with the centre at the origin.

(ii) Now, $zf(z) = z.\frac{1}{(z^2+2^2)(z^2+3^2)} \rightarrow 0\ as|z| \rightarrow \infty$

(iii) The poles are given by $z + 2^2 = 0, z^2 + 3^2 = 0 \quad \therefore z = \pm 2i\ , z = \pm 3i.$ Of these

$z = 2i, z = 3i$ lie in the upper half of the z plane.

(iv) Residue (at $z = 2i$) $= \lim_{z \to ai} (z - 2i).\frac{1}{(z-2i)(z+2i)(z^2+3^2)} = \frac{1}{4i(3-2)}$

Similarly, Residue $(at\ z = 3i) = \lim_{z \to 3i} (z - 3i).\frac{1}{(z^2+2^2)(z-3i)(z+3i)}$

$$= \frac{1}{6i} \cdot \frac{1}{2^2-3^2)}$$

(v) $\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{(\mathrm{x}^2+4)(\mathrm{x}^2+9))} = \quad 2\pi i \left[-\frac{1}{4i(3-2)} + \frac{1}{6i} \cdot \frac{1}{2^2-3^2)}\right]$

$$= \frac{\pi}{30}$$

Q.6)

(a) Evaluate $\int_c \frac{dz}{z^3(z+4)}$ **where C is a circle** **[6]**

i. $|z| = 2$

ii. $|z-3| = 2$

<u>Solution:</u>

(i)

The poles are given by $z^3(z+4) = 0$

$\therefore z = 0$ is a pole of order 3 and $z = -4$ is a simple pole.

$|z| = 2$ is a circle with centre at the origin and radius 2 . Hence $z = 0$ lies inside C and $z = -4$ lies outside.

$$\text{Residue at } z = 0 = \lim_{z\to 0} \frac{1}{2!} \cdot \frac{d^2}{dz^2}\left[z^3 \cdot \frac{1}{z^3(z+4)}\right]$$

$$= \lim_{z\to 0} \frac{1}{2} \cdot \frac{d^2}{dz^2}\left(\frac{1}{z+4}\right) = \lim_{z\to 0} \frac{1}{2} \cdot \frac{d}{dz}\left(-\frac{1}{z+4}\right)$$

$$= \lim_{z\to 0} \frac{1}{2} \cdot \frac{2}{(z+4)^3} = \frac{1}{64}$$

$$\therefore \int_c \frac{dz}{z^3(z+4)} = 2\pi i \left(\frac{1}{64}\right) = \frac{\pi i}{32}$$

iii. $|z-3|=2$

$\therefore$ Centre at (3,0) and radius is 2.

As mentioned above $z^3(z+4)=0$ gives $z=0$ and $z=-4$

So $z(3,0)$ lies inside the c and $z(0,-4)$ lies outside the c

Hence we put $f(z)$ is analytic in c

Residue at $z=3 = \lim_{z\to 3} \frac{1}{2!}.\frac{d^2}{dz^2}\left[z^3.\frac{1}{z^3(z+4)}\right]$

$$= \lim_{z\to 3} \frac{1}{2}.\frac{d^2}{dz^2}\left(\frac{1}{z+4}\right) = \lim_{z\to 3} \frac{1}{2}.\frac{d}{dz}\left(-\frac{1}{z+4}\right)$$

$$= \lim_{z\to 3} \frac{1}{2}.\frac{2}{(z+4)^3} = \frac{1}{343}$$

$$\therefore \int_c \frac{dz}{z^3(z+4)} = 2\pi i\left(\frac{1}{343}\right) = \frac{2\pi i}{343}$$

(b) Two unbiased dice are thrown three times, using Binomial distribution find the probabilities that the sum nine would be obtained **[6]**

i. Once

ii. Twice

<u>Solution</u>

When two unbiased dice are thrown the chances that sum 9 are obtained are:[(6,3); (3,6); (5,4); (4,5)]

So there are 4 chances to obtain sum 9 .

$\therefore p = \frac{4}{36}, q = 1 - \frac{4}{36}, n = 3$

(i) Once

By Binomial Distribution

$P(X = x) = {}^{n}C_x p^x q^{n-x}$

$P(X = 1) = {}^{3}C_1 \left(\frac{4}{36}\right)^1 \left(1 - \frac{4}{36}\right)^{3-2} = {}^{3}C_1 \left(\frac{4}{36}\right)^1 \left(\frac{32}{36}\right)^2 = 0.26$

(ii) Twice

By Binomial Distribution

$P(X = x) = {}^{n}C_x p^x q^{n-x}$

$P(X = 1) = {}^{3}C_2 \left(\frac{4}{36}\right)^2 \left(1 - \frac{4}{36}\right)^1 = {}^{3}C_1 \left(\frac{4}{36}\right)^2 \left(\frac{32}{36}\right)^1 = 0.3$

(c) For the Following data

X	**100**	**110**	**120**	**130**	**140**	**150**	**160**	**170**	**180**	**190**
Y	**45**	**51**	**54**	**61**	**66**	**70**	**74**	**78**	**85**	**89**

Find the coefficients of regression b_{xy} & b_{yx} and the corfficient of correlation (r) . **[8]**

Solution:

Calculation of b_{yx} ,b_{xy} etc.

Sr no	Dx			dy			$d_x d_y$
	X	$X-150$	d_x^2	Y	X-70	d_y^2	
1	100	-50	2500	45	-25	625	1250
2	110	-40	1 600	51	-19	361	760
3	120	-30	900	54	-16	256	480
4	130	-20	400	61	-9	81	180
5	140	-10	100	66	-4	16	40
6	150	00	000	70	0	0	00
7	160	10	100	74	4	16	40
8	170	20	400	78	8	64	160
9	180	30	900	85	15	225	450
10	190	40	1600	89	19	361	760
$N = 10$	50 8500			-27 2005			4120

$$\overline{X} = A + \sum \frac{dx}{N} = 150 - \frac{50}{10} = 145; \quad \overline{Y} = B + \sum \frac{dy}{N} = 70 - \frac{27}{10} = 67.3$$

$$b_{yx} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{N}}{\sum d_x^2 - \frac{\sum (d_x)^2}{N}} = \frac{4120 - \frac{(-50)(-27)}{10}}{8500 - \frac{(-50)^2}{10}} = \frac{4120 - 135}{8500 - 250} = \frac{3985}{8250} = 0.483$$

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{N}}{\sum d_y^2 - \frac{\sum (d_y)^2}{N}} = \frac{4120 - \frac{(-50)(-27)}{10}}{8500 - \frac{(-27)^2}{10}} = \frac{4120 - 135}{2005 - 72.9} = \frac{3985}{1932.1} = 2.06$$

The line of regression of Y on X is

$$Y - \overline{Y} = b_{yx}(X - \overline{X}$$

$$\therefore Y - 67.3 = 0.483(X - 145) \qquad \therefore Y = 0.483X - 2.735$$

The coefficient of correlation (r) is

$$r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0.483 \times 2.06} = 0.9975$$

Hence, $b_{yx} = 0.483$, $b_{xy} = 2.06$, $r = 0.9975$.

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