MATHEMATICS SOLUTION

CBCGS (DEC – 2019) SEM – 3

BRANCH – COMPUTER ENGINEERING

Q1. a) If $L\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$. Find $L\{\omega t\cos t\omega t + \sin\omega t\}$ **[5]**

Soln.: $L\{\omega t\cos t\omega t + \sin\omega t\}$

$L\{\omega t\cos\omega t\} + L\{\sin\omega t\}$

$\omega L\{t\cos\omega t\} + L\{\sin\omega t\}$ -----(i)

Finding $L\{t\cos\omega t\}$;

$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$

$L[t\cos\omega t] = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{s^2+\omega^2}$

$L[\sin\omega t] = \frac{1}{s^2+\omega^2}$

Substituting in (i), we get;

$\mathbf{L\{\omega t\cos t\omega t + \sin\omega t\} = \omega\frac{s^2-\omega^2}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}}$

Q1. b) If $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic, find a, b, c and d. [5]

Soln.: We have $f(z) = u + iv$ and $u = (x^2 + axy + by^2)$; $v = (cx^2 + dxy + y^2)$

$\therefore u_x = 2x + ay$ $\quad u_y = ax + 2by$

$v_x = 2cx + dy$ $\quad v_y = dx + 2y$

By CR equation,

$u_x = v_y$

$2x + ay = dx + 2y$

On comparing the coefficients,

$a = 2$ and $d = 2$

Also, $u_y = -v_x$

$$ax + 2by = -(2cx + dy)$$

$$2x + 2by = -2cx - 4y$$

On comparing the coefficients,

$c = -1$ and $b = -2$

Ans: $\mathbf{a = 2, b = -2, c = -1}$ and $\mathbf{d = 2}$

Q1. c) Find the Fourier series of expansion of $\mathbf{f(x) = x^3\ (-\pi,\ \pi)}$ [5]

Soln. : $f(x) = x^3$ is an odd function as $f(x) = f(-x) = -f(x)$

Therefore in the range $(-\pi,\ \pi)$, $a_0 = a_1 = 0$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - \left\{3x^2\left(-\frac{\sin nx}{n^2}\right)\right\} + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\pi^3\left(-\frac{\cos n\pi}{n}\right) - \left\{3\pi^2\left(-\frac{\sin n\pi}{n^2}\right)\right\} + 6\pi\left(\frac{\cos n\pi}{n^3}\right) - 6\left(\frac{\sin n\pi}{n^4}\right) - \{0 - 0 + 0 - 0\}\right]$$

$$b_n = \frac{2}{\pi}\left[\pi^3\left(-\frac{(-1)^n}{n}\right) + 6\pi\left(\frac{(-1)^n}{n^3}\right)\right]$$

$$b_n = 2\left[-\pi^2\left(\frac{-1^n}{n}\right) + 6\left(\frac{-1^n}{n^3}\right)\right]$$

$$b_n = (-1)^n\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]$$

Fourier series for the given function is given as: $f(x) = \sum_{x=1}^{x=\infty} b_n \sin nx$

$$\mathbf{f(x) = \sum_{x=1}^{x=\infty}(-1)^n\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]\sin nx}$$

Q1. d) If the two regression equations are $4x - 5y + 33 = 0, 20x - 9y - 107 = 0$. Find:

i) The mean values of x and y
ii) The Correlation Coefficient
iii) Standard Deviation of y if variance of x is 9 **[5]**

Soln.:

i) Solving the equations simultaneously,

$$4x - 5y = -33$$
$$20x - 9y = 107$$

We get $\bar{x}$= 13 and $\bar{y}$=17

ii) Suppose the second equation represents the line of regression of X on Y

$$20x = 9y + 107$$

$$\therefore b_{xy} = \frac{9}{20}$$

Suppose the first equation represents the line of regression of X on Y

$$5y = 4x + 33$$

$$\therefore b_{yx} = \frac{4}{5}$$

$$r = \sqrt{b_{xy}.b_{yx}} = \sqrt{\frac{9}{20}.\frac{4}{5}} = \mathbf{0.6}$$

iii) $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$\frac{4}{5} = 0.6(\frac{\sigma_y}{3})$ [Standard devn = $\sqrt{\text{variance}}$]

$\boldsymbol{\sigma_y = 4}$

Q2. a) Show that the function is harmonic and find the harmonic conjugate. **[6]**

$$\mathbf{u = \cos x\ \cosh y - 2xy}$$

Soln.: Given: $u = \cos x\ \cosh y - 2xy$

Partially double differentiating wrt x and y.

$u_x = -\cosh y \sin x - 2y$

$u_x^2 = -\cosh y \cos x$

$u_y = \cos x \sinh y - 2x$

$u_y^2 = \cos x \cosh y$

By Laplace's equation,

$u_x^2 + u_y^2 = 0$

$-\cosh y \cos x + \cos x \cosh y = 0 = RHS$

Thus, the function is harmonic

$-\int u_y dx = -\int (\cos x \sinh y - 2x)dx$

= $-\sin x \sinh y + x^2$

Integrating terms in u_x free from x

$\int -2y\, dy = -y^2$

$\therefore$ $\mathbf{v = \sin x \sinh y + x^2 - y^2 + c}$

Q2. b) Evaluate $\int_0^\infty e^{-t}(\int_0^t u^2 \sinh u \cosh u\, du)dt$ using Laplace Transform **[6]**

Soln.: $L[\sinh u \cosh u] = \frac{1}{2}[2 \sinh u \cosh u] = \frac{1}{2} L[\sinh 2u] = \frac{1}{2}[\frac{2}{s^2-2^2}]$

$L[u^2 \sinh u \cosh u] = \frac{d^2}{ds^2}(\frac{1}{s^2-4})$

$\frac{d}{ds}\left(-\frac{2s}{s^2-4}\right) = \frac{2(3s^2+4)}{(s^2-4)^3}$

$L\left[\int_0^t u^2 \sinh u \cosh u\, du\right] = \frac{2(s^2+4)}{s(s^2-4)^3}$

$\therefore \int_0^\infty e^{-st}(\int_0^t u^2 \sinh u \cosh u\, du)dt = \frac{2(3s^2+4)}{s(s^2-4)^3}$

Put s=1, we get

$$\int_0^\infty \mathbf{e^{-t}}(\int_0^t \mathbf{u^2 \sinh u \cosh u\, du})\mathbf{dt} = \frac{\mathbf{2(3+4)}}{\mathbf{1(1-4)^3}} = -\frac{\mathbf{14}}{\mathbf{27}}$$

Q2. c) Find the Fourier Series expansion of $f(x) = \begin{cases} x\,;\ -1 < x < 0 \\ x + 2\,;\ 0 < x < 1 \end{cases}$ **[8]**

Soln.: Fourier series for f(x) is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$a_0 = \frac{1}{2l}\int_{-1}^{1} f(x)dx = \frac{1}{2}[\int_{-1}^{0}(x+2)dx + \int_{0}^{1} xdx]$

$a_0 = \frac{1}{2}\left[\left(\frac{x^2+4x}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{0}^{1}\right]$

$a_0 = \frac{1}{2}\left[\left(0 + \frac{3}{2}\right) + \left(\frac{1}{2}\right)\right] = 1$

$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos(\frac{n\pi x}{l})dx = [\int_{-1}^{0}(x+2)\cos(n\pi x)dx + \int_{0}^{1} x\cos(n\pi x)dx]$

$a_n = 1\left[\left(\frac{(x+2)}{n\pi}\sin n\pi x + \frac{2(\cos n\pi x)}{n^2\pi^2}\right)_{-1}^{0} + \left(\frac{x}{n\pi}\sin n\pi x + \frac{1(\cos n\pi x)}{n^2\pi^2}\right)_{0}^{1}\right]$

$a_n = 1\left[\left(0 + \frac{2}{n^2\pi^2} - 0 - \frac{2}{n^2\pi^2}\right) + \left(0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{(-1)^n}{n^2\pi^2}\right)\right] = 0$

$b_n = \frac{1}{l}\int_{-1}^{1} f(x)\sin(\frac{n\pi x}{l})dx = [\int_{-1}^{0}(x+2)\sin(n\pi x)dx + \int_{0}^{1} x\sin(n\pi x)dx]$

$b_n = 1\left[\left(-\frac{(x+2)}{n\pi}\cos n\pi x + \frac{2(\sin n\pi x)}{n^2\pi^2}\right)_{-1}^{0} + \left(-\frac{x}{n\pi}\cos n\pi x + \frac{1(\sin n\pi x)}{n^2\pi^2}\right)_{0}^{1}\right]$

$b_n = 1\left[\left(-\frac{2}{n\pi} - 0 + \frac{(-1)^n}{n\pi} - 0\right) + \left(-\frac{(-1)^n}{n\pi} - 0\right)\right] = -\frac{2}{n\pi}$

Substituting the values in expansion,

$$f(x) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$$

Q3.a) Find the Analytic function f(z)=u+iv if $u - v = e^x(\cos y - \sin y)$ **[6]**

Soln.: Let $U = u - v = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y) = \phi 1(x)$

$U_y = e^x(-\sin y - \cos y) = -e^x(\sin y + \cos y) = \phi 2(x)$

$\therefore\ (1+i)f'(z) = U_x - iU_y = \phi 1(z,0) - i\phi 2(z,0)$

$\therefore\ (1+i)f(z) = \int [e^z + ie^z]dz = (1+i)\int e^z dz = (1+i)e^z + C$

$\therefore\ \mathbf{f(z) = e^z + C}$

Q3.b) Find Inverse Z transform of $\frac{5z}{(2z-1)(z-3)}$ $\quad \frac{1}{2} < |z| < 3$ [6]

Soln.: We have, $F(Z) = \frac{5z}{(2z-1)(z-3)}$

Applying Partial fractions;

$$\frac{z}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$$

$$\frac{z}{(2z-1)(z-3)} = \frac{A(z-3) + B(2z-1)}{(2z-1)(z-3)}$$

Comparing the coefficients on both the sides,

$$1 = A + 2B \text{ and } 0 = 3A + B$$

Solving the equations simultaneously,

$$A = -\frac{1}{5} \text{ and } B = \frac{3}{5}$$

$$\frac{5z}{(2z-1)(z-3)} = \left[\frac{3}{z-3} - \frac{1}{2z-1}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[\frac{3}{z-3} - \frac{1}{2(z-\frac{1}{2})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\frac{3}{3(1-\frac{z}{3})} - \frac{1}{2z(1-\frac{1}{2z})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\frac{1}{1(1-\frac{z}{3})} - \frac{1}{2z(1-\frac{1}{2z})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{2z}\right)^{-1}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots+\left(\frac{z}{3}\right)^n\right) - \frac{1}{2z}\left(1+\frac{1}{2z}+\frac{1}{4z^2}+\cdots+\frac{1}{(2z)^n}\right)\right]$$

Coefficient of z^n in first series $= -3$

Put n=-k

$z^{-k} = -3$ k > =0

Coefficient of z^{-n} in second series $= \frac{1}{2^n}$

Put n=k

$z^{-k} = \frac{1}{2^k}$ k > = 0

$$\mathbf{Z^{-1}[F(Z)] = -3 + \frac{1}{2^k}; k >= 0}$$

Q3. c) Solve the differential equation using Laplace Transform: **[8]**

$$(D^2 - 2D + 1)y = e^t, y(0) = 2 \text{ and } y'(0) = -1$$

Soln.: Let $L(y) = \bar{y}$, then taking Laplace transform on both sides,

$L(y'') - 2L(y') + L(y) = L(e^t)$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s + 1$

and $L(e^t) = \frac{1}{s-1}$

$\therefore$ the equation becomes,

$s^2\bar{y} - 2s + 1 - 2(s\bar{y} - 2) + \bar{y} = \frac{1}{s-1}$

$\Rightarrow s^2\bar{y} - 2s + 1 - 2s\bar{y} + 4 + \bar{y} = \frac{1}{s-1}$

$\Rightarrow \bar{y}(s^2 - 2s + 1) = \frac{1}{s-1} + 2s - 5 \quad \Rightarrow \quad \bar{y}(s-1)^2 = \frac{1}{s-1} + 2s - 5$

$\Rightarrow \bar{y} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{1}{(s-1)^3} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2\left[\frac{(s-1)}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - \frac{5}{(s-1)^2} \Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2\left[\frac{1}{(s-1)} + \frac{1}{(s-1)^2}\right] - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2[e^t] - \frac{3}{(s-1)^2}$

$\Rightarrow \frac{e^t t^2}{2} + 2[e^t] - \frac{3e^t}{(s)^2} \Rightarrow \frac{e^t t^2}{2} + 2[e^t] - 3te^t$

Ans : $\frac{e^t t^2}{2} + 2[e^t] - \frac{3e^t}{(s)^2} \Rightarrow \frac{e^t t^2}{2} + 2[e^t] - 3te^t$

Q4. a) Find the Complex form of Fourier Series for f(x)= cos ax ; (-π,π) **[6]**

Soln.: We have $\cos ax = (e^{aix} + e^{-aix}) / 2$

Complex form of f(x) is given by $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$

For e^{aix}:

$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{aix}e^{-inx}\,dx$

$$= \frac{1}{2\pi}\left[\frac{e^{(ai-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(ai-in)}(e^{ai\pi}e^{-inx} - e^{-ai\pi}e^{in\pi})$$

We know $e^{in\pi} = (-1)^n$

∴

$$C_n = \frac{1}{2\pi(ai-in)}\left(e^{ai\pi}(-1)^n - e^{-ai\pi}(-1)^n\right) = \frac{(-1)^n}{2\pi(ai-in)}(e^{ai\pi} - e^{-ai\pi})$$

Multiply and divide by 2,

$C_n = \frac{(-1)^n}{\pi(ai-in)}\left(\frac{e^{ai\pi}-e^{-ai\pi}}{2}\right) = \frac{(-1)^n}{\pi(ai-in)}(\sinh ai\pi) = \frac{i(-1)^n}{\pi(ai-in)}\sin a\pi$

$C_n = \frac{i(-1)^n}{\pi(ai-in)}\sin a\pi \,.\, \frac{ai+in}{ai+in} = \frac{i(-1)^n(ai+in)}{\pi(-a^2+n^2)}\sin a\pi = \frac{(-1)^n(a+n)}{\pi(a^2-n^2)}\sin a\pi$

∴

$$f(x) = \frac{\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx}$$

Similarly for e^{-aix}, we get

$$f(x) = \frac{\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a-n)}{(a^2-n^2)}e^{inx}$$

∴

$$\cos ax = (e^{aix} + e^{-aix}) / 2$$

$$\cos ax = \frac{\sin a\pi}{2\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx} + \frac{\sin a\pi}{2\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a-n)}{(a^2-n^2)}e^{inx}$$

$$\therefore \mathbf{\cos ax} = \frac{\mathbf{a\sin a\pi}}{\boldsymbol{\pi}}\sum_{-\infty}^{\infty}\frac{(-\mathbf{1})^{\mathbf{n}}}{(\mathbf{a^2-n^2})}\mathbf{e^{inx}}$$

Q4. b) Find the Spearman's Rank correlation coefficient between X and Y. [6]

X	68	64	75	50	64	80	75	40	55	64
Y	62	58	68	45	81	60	68	48	50	70

Soln.:

Sr No.	X	R1	Y	R2	D = (R1 − R2)^2
1	68	7	62	6	1
2	64	5	58	4	1
3	75	8.5	68	7.5	1
4	50	2	45	1	1
5	64	5	81	10	25
6	80	8.5	60	5	12.25
7	75	10	68	7.5	6.25
8	40	1	48	2	1
9	55	3	50	3	0
10	64	5	70	9	16
N=10					∑=64.5

m1=3, m2=2 and m3=3

$$R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(m1^3 - m1) + \frac{1}{12}(m2^3 - m2) + \frac{1}{12}(m3^3 - m3)\right]}{N^3 - N}$$

$$R = 1 - \frac{6\left[64.5 + \frac{1}{12}(24) + \frac{1}{12}(6) + \frac{1}{12}(6)\right]}{990}$$

Ans : $\mathbf{R = 0.9327}$

Q4.c) Find the inverse Laplace transform of

i) $\frac{s-1}{s^2+2s+2}$ **ii)** $\frac{e^{-\pi s}}{s^2(s^2+1)}$ **[8]**

Soln.:

i) $L^{-1}\left[\frac{s-1}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+1}\right]$

$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$

$= e^{-t}L^{-1}\left[\frac{(s)}{(s)^2+1} - \frac{1}{(s)^2+1}\right]$

$== e^{-t}[\cos t - \sin t]$

ii) $L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right]$

Here $\phi(s) = \frac{1}{s^2(s^2+1)}$ and $a = \pi$

$\therefore L^{-1}[\phi(s)] = L^{-1}[\frac{1}{s^2(s^2+1)}]$

Applying convolution theorem,

Let $\Phi_1(s) = \frac{1}{s^2}$; $\Phi_2(s) = \frac{1}{s^2+1}$

$\therefore\ L^{-1}[\Phi_1(s)] = t$; $L^{-1}[\Phi_2(s)] = \sin t$

$\therefore L^{-1}[\phi(s)] = \int_0^t \sin t\ .(t-u)du$

$=\sin t\ \left[tu - \frac{u^2}{2}\right]_0^t = \sin t(t^2 - \frac{t^2}{2})$

$\therefore\ L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right] = \mathbf{f(t-a)H(t-a)}$

$\therefore$

$$\text{Ans: } L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right] = \mathbf{\sin(t-\pi)\left[(t-\pi)^2 - \frac{(t-\pi)^2}{2}\right]H(t-\pi)}$$

Q5. a) Find the $Z\{f(k)\} = 4^k, k < 0$ **[6]**

$= \ 3^k, k \geq 0$

Soln.: By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$

$\therefore\ Z\{f(k)\} = \sum_{k=-\infty}^{-1} 4^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$

Putting k = -n in the first series, we get

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 4^{-n}z^{n} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$Z\{f(k)\} = \left[\frac{z}{4} + \frac{z^2}{4^2} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right]$$

$$Z\{f(k)\} = \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right]$$

$$Z\{f(k)\} = \left[\frac{4}{4-z}\right] + \left[\frac{z}{z-3}\right]$$

ROC is $3 < |z| < 4$

Q5.b) Show that {cos x, cos 2x, cos 3x, … } is orthogonal set over the interval $[0,2\pi]$. Construct the corresponding orthonormal set. **[6]**

Soln.: We have $f_n(x) = \cos nx; n = 1,2,3$

Therefore, $\int_{-\pi}^{\pi} f_m(x)f_n(x)dx \Rightarrow\ \int_{-\pi}^{\pi} \cos m\, x. \cos nxdx$

$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)\, x + \cos(m-n)\, xdx \Rightarrow \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)}{m-n}\right]_{-\pi}^{\pi}$

Now two cases arises:

i) When m≠n: $\frac{1}{2}\left[\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)}{m-n}\right\} - \left\{-\frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)\pi}{m-n}\right\}\right] = 0$

ii) When m=n: $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2} dx$

$\frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \frac{1}{2}[\pi + 0 + \pi - 0] = \pi \neq 0$

Therefore the functions are orthogonal in $[-\pi,\pi]$

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \pi$$

Dividing the equation by π,

$\Rightarrow \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} f(x). \frac{1}{\sqrt{\pi}} f(x) dx = 1$

This is obviously an orthonormal set where $\phi(x) = \frac{1}{\sqrt{\pi}} \cos nx$

Thus the required orthonormal set is $\frac{1}{\sqrt{\pi}} \cos x$, $\frac{1}{\sqrt{\pi}} \cos 2x$, $\frac{1}{\sqrt{\pi}} \cos 3x$, ...

Q5. c) Find the bilinear transformation which maps the points z= 1, i, -1 into the points w = i, 0, -i. Hence find the image of |z|<1 [8]

Soln.: Let the transformation be $w = \frac{az+b}{cz+d}$ -----(1)

Putting the given values of z and w, we get,

$i = \frac{a+b}{c+d}$; $0 = \frac{ai+b}{ci+d}$; $-1 = \frac{-a+b}{-c+d}$

From these equalities, we get,

(a+b) - i(c+d) = 0 -----(2) b+ia=0 -----(3)

(-a-b) + i(-c+d) = 0 -----(4)

From 2 and 4 we get c=b/i

Subtracting 4 from 2, we get 2a – 2id =0. ∴d=-ia

Putting the values b=-ia, c=-a and d=-ia in (1) we get,

$w = \frac{az-ia}{-az-ia} = \frac{z-i}{-z-i}$

∴ **$w = \frac{i-z}{i+z}$ is the required bilinear transformation.**

∴ wi + wz = i - z

w – i = -z(1 + w)

Further, |z| < 1 is mapped onto the region

$$\left|\frac{i(1-w)}{1+w}\right| < 1$$

∴ |1-w| < |1+w| [|i|=1]

|(1-u)-iv| < |(1+u)+iv|

$\therefore (1-u)^2 + v^2 < (1+u)^2 + v^2$

$-4u<0 \Rightarrow$ **u>0**

Q6. a) Fit a straight line to the given data **[6]**

X	**10**	**12**	**15**	**23**	**20**
Y	**14**	**17**	**23**	**25**	**21**

Soln.:

x	y	x^2	xy
10	14	100	140
12	17	144	204
15	23	225	345
23	25	529	575
20	31	400	620
$\sum x=80$	$\sum y=110$	$\sum x^2 = 1398$	$\sum xy=1884$

Let the equation be y=a+bx

The normal equations are

$\Sigma y = Na + b\Sigma x$ $\therefore 110=5a+80b$

And $\Sigma xy = a\Sigma x + b\Sigma x^2$ $\therefore 1884=80a+1398b$

Solving the equations simultaneously,

a=306/59 and b=62/59

Q6.b) Find the Inverse Laplace Transform using convolution theorem **[6]**

$$\frac{\mathbf{1}}{\mathbf{(s-2)^3(s+3)}}$$

Soln.: Let $\Phi_1(s) = \frac{1}{s+3}$; $\Phi_2(s) = \frac{1}{(s-2)^3}$

$\therefore L^{-1}[\Phi_1(s)] = e^{-3t}$; $L^{-1}[\Phi_2(s)] = e^{2t}L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}.\frac{t^2}{2}$

$\therefore L^{-1}[\phi(s)] = \int_0^t e^{-3u}.e^{2(t-u)}.\frac{(t-u)^2}{2}du$

$= \int_0^t e^{(2t-5u)}.\frac{(t-u)^2}{2}du$

$= e^{2t}\left[\frac{(t-u)^2}{2}\left(\frac{-e^{-5u}}{5}\right) - (t-u)\left(\frac{e^{-5u}}{25}\right) + \left(\frac{e^{-5u}}{125}\right)\right]_0^t$

$= e^{2t}\left[\left\{0 - 0 - \left(\frac{e^{-5t}}{125}\right)\right\} - \left\{\frac{(t)^2}{2}\left(\frac{-1}{5}\right) - (t)\left(\frac{1}{25}\right) + \left(\frac{1}{125}\right)\right\}\right] = e^{2t}[-\left(\frac{e^{-5t}}{125}\right) + \frac{t^2}{10} + \frac{t}{25} + \frac{1}{125}]$

$$\textbf{Ans}: e^{2t}\left[\left(\frac{t^2}{10} + \frac{t}{25} + \frac{1}{125}\right) - \frac{e^{-5t}}{125}\right]$$

Q6.c) Find Half Range Cosine Series for f(x)=sin x in (0,π) and hence deduce that [8]

$$\frac{\pi^2 - 8}{16} = \frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + \cdots$$

Soln.: Let $f(x) = a_0 + \sum a_n \cos nx$

$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}[-\cos x]_0^\pi$

$\therefore a_0 = -\frac{1}{\pi}[-1-1] = \frac{2}{\pi}$

$a_n = \frac{2}{\pi}\left[\int_0^\pi f(x)\cos nx\, dx\right] = \frac{2}{\pi}\left[\int_0^\pi \sin x \cos nx\, dx\right]$

$a_n = \frac{2}{2\pi}\left[\int_0^\pi \sin(1+n)x + \sin(1-n)x\right]dx$

$a_n = \frac{1}{\pi}\left[\frac{-\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{1-n} - \left(-\frac{1}{1+n} - \frac{1}{1-n}\right)\right]$

$a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{(1+n)} - \frac{\cos n\pi}{n-1} + \left(\frac{1}{1+n} + \frac{1}{n-1}\right)\right]$

$a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{(1+n)}\left(\frac{2}{n^2-1}\right) - \frac{2}{n^2-1}\right] = -\frac{2}{\pi(n^2-1)}[(-1)^n + 1]$

= 0 if n is odd and n is not = 1

$\therefore a_n = -\frac{4}{\pi(n^2-1)}$ if n is even

$\therefore$ If n =1 , we get

$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\, dx = \frac{1}{\pi}\int_0^\pi \sin 2x dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi$

$a_1 = \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0$

$\therefore$

$$\textbf{Ans: } f(x) = \sin\ x = \frac{2}{\pi} - \sum \frac{4}{\pi(n^2-1)} \cos nx$$
