COMPUTER ENGINEERING

APPLIED MATHEMATICS -3

(CBCGS - MAY 2018)

Q1.a) Find the Laplace transform of $e^{-2t} t\cos t$ **[5]**

Sol : $L[\cos t] = \frac{s}{s^2+1}$ $\left\{\because L[\cos at]=\frac{s}{s^2+a^2}\right\}$

$\Rightarrow L[t \cos t] = (-1)\left[\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right]$ $\left\{\text{By } \frac{u}{v} \text{ rule of differentiation}\right\}$

$\Rightarrow L[t \cos t] = -\left[\frac{(s^2+1)-2s^2}{(s^2+1)^2}\right] \Rightarrow \left[\frac{(s^2-1)}{(s^2+1)^2}\right]$

$\Rightarrow L[e^{-2t} t \cos t] = \left[\frac{(s+2)^2-1}{((s+2)^2+1)^2}\right]$ $\{L[e^{-at} f(t)] = \Phi(s+a)\}$

$\Rightarrow L[e^{-2t} t \cos t] = \left[\frac{s^2+4s-3}{(s^2+4s+5)^2}\right]$

Ans : $L[e^{-2t} t \cos t] = \left[\frac{s^2+4s-3}{(s^2+4s+5)^2}\right]$

Q1.b) Find the inverse Laplace transform of $\frac{3s+7}{s^2-2s-3}$ **[5]**

Sol : Adjusting the numerator and denominator

$\Rightarrow \frac{3(s-1)+10}{(s-1)^2-2^2}$

Splitting the terms;

$\Rightarrow 3L^{-1}\left[\frac{(s-1)}{(s-1)^2-2^2}\right] + 10L^{-1}\left[\frac{1}{(s-1)^2-2^2}\right]$

$\Rightarrow 3e^t L^{-1}\left[\frac{s}{s^2-2^2}\right] + 10e^t L^{-1}\left[\frac{1}{s^2-2^2}\right]$ $\{\because \Phi(s+a) = e^{-at} L[f(t)]\}$

$\Rightarrow 3e^t \cosh 2t + \frac{10}{2} e^t \sinh 2t$ $\left\{\because L\left[\frac{s}{s^2-a^2}\right]=\cosh at, L\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}\sinh at\right\}$

$\Rightarrow e^t(3\cosh 2t+5\sinh 2t)$

Ans : $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = e^t(3\cosh 2t+5\sinh 2t)$

Q1.c) Determine whether the function $f(z) = (x^3+3xy^2-3x) + i(3x^2y-y^3+3y)$ is analytic and if so, find its derivative. [5]

Sol : Given $f(z) = (x^3+3xy^2-3x) + i(3x^2y-y^3+3y)$

Comparing real and imaginary parts, we get

$u = (x^3+3xy^2-3x)$; $v = (3x^2y-y^3+3y)$

Differentiating u partially w.r.t x and y,

$u_x = 3x^2+3y^2-3$; $u_y = 6xy$

Differentiating v partially w.r.t x and y,

$v_x = 6xy$; $v_y = 3x^2-3y^2+3$

$\because$ CR equations are not satisfied $\{u_x \neq v_y ; u_y \neq -v_x\}$

Therefore the function is not analytic and thus its derivative does not exists.

Q1.d) Find the Fourier series for $f(x) = x^2$ in the interval $(-\pi,\pi)$ [5]

Sol : $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$

Fourier transform for even function is given by :

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ -----(i)

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx \Rightarrow \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \Rightarrow \frac{1}{3\pi}(\pi^3-0)$

$\Rightarrow a_0 = \frac{\pi^2}{3}$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\left\{\frac{\sin nx}{n}\right\}-2x\left\{\frac{-\cos nx}{n^2}\right\}+2\left\{\frac{-\sin nx}{n^3}\right\}\right]_0^\pi$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{0-2\pi\left(\frac{-\cos n\pi}{n^2}\right)+0\right\}-\{0-0+0\}\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{2\pi\left(\frac{\cos n\pi}{n^2}\right)\right\}\right] \Rightarrow a_n = \frac{4}{n^2}(-1)^n \qquad \{\because \cos n\pi=(-1)^n\}$$

Resubstituting the values in (i)

$$\textbf{Ans}: x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

Q2.a) Evaluate $\int_0^\infty\left(\frac{\sin 2t+\sin 3t}{te^t}\right)dt = \frac{3\pi}{4}$ **[6]**

Sol : LHS :

$$L(\sin 2t+\sin 3t) = \frac{2}{s^2+4} + \frac{3}{s^2+9} \qquad \left\{\because L[\sin at]=\frac{a}{s^2+a^2}\right\}$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \int_s^\infty\frac{2}{s^2+4}ds + \int_s^\infty\frac{3}{s^2+9}ds$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\left\{\tan^{-1}(\infty)+\tan^{-1}(\infty)\right\}-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\}-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\int_0^\infty e^{-st}L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

On Putting s=1,

$$\int_0^\infty e^{-t}L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$\int_0^\infty e^{-t} L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}\right)\right] \qquad \left\{\because \tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)\right\}$$

$$\Rightarrow\left[\pi - \tan^{-1}\left(\frac{5}{5}\right)\right]$$

$$\Rightarrow\left[\pi-\frac{\pi}{4}\right]$$

$$\Rightarrow\left[\frac{3\pi}{4}\right]$$

=RHS

Hence proved.

Q2.b) Find the Z-transform of $\left\{\left(\frac{1}{4}\right)^{|k|}\right\}$ [6]

Sol : $F(Z) = \begin{cases}\left(\frac{1}{4}\right)^{k} ;k\geq 0\\ \left(\frac{1}{4}\right)^{-k} ;k<0\end{cases}$

The equation can be expressed as : $\sum_{-\infty}^{\infty} F(z).z^{-k}$

$$\Rightarrow\sum_{-\infty}^{-1}\left(\frac{1}{4}\right)^{-k} z^{-k}+\sum_{0}^{\infty}\left(\frac{1}{4}\right)^{k} z^{-k}$$

$$\Rightarrow\left[...+\left(\frac{z}{4}\right)^{3}+\left(\frac{z}{4}\right)^{2}+\left(\frac{z}{4}\right)^{1}\right] + \left[1+\left(\frac{1}{4z}\right)^{1}+\left(\frac{1}{4z}\right)^{2}+\left(\frac{1}{4z}\right)^{3}+...\right]$$

The above two series are sum of infinite GP whose sum is given as: $\frac{a}{1-r}$

Where a = 1st term, r is the common ratio between the terms.

$$\Rightarrow\left(\frac{z}{4}\right)\left[...+\left(\frac{z}{4}\right)^{2}+\left(\frac{z}{4}\right)^{1}+1\right] + \left[1+\left(\frac{1}{4z}\right)^{1}+\left(\frac{1}{4z}\right)^{2}+\left(\frac{1}{4z}\right)^{3}+...\right]$$

$$\Rightarrow\left(\frac{z}{4}\right)\left[\frac{1}{1-\frac{z}{4}}\right] + \left[\frac{1}{1-\left(\frac{1}{4z}\right)}\right] \qquad , \quad \left|\frac{z}{4}\right| < 1 \text{ and } \left|\frac{1}{4z}\right| < 1$$

Ans : $Z\{f(k)\} = \left(\frac{z}{4}\right)\left[\frac{1}{1-\frac{z}{4}}\right] + \left[\frac{1}{1-\left(\frac{1}{4z}\right)}\right]; \left|\frac{1}{4}\right| < z < 4$

Q2.c) Show that the function v = $e^x(x \sin y + y \cos y)$ **is harmonic function. Find its harmonic conjugate and corresponding analytic function. [8]**

Sol : $v = e^x(x \sin y + y \cos y)$

$v = e^x x \sin y + e^x y \cos y$

Differentiating partially wrt. x and y twice,

$v_x = e^x(x \sin y + y \cos y) + e^x \sin y$

$v_y = e^x(x \cos y + \cos y - y \sin y)$

$v_x^2 = e^x(x \sin y + y \cos y) + e^x \sin y + e^x \sin y$ -----(i)

$v_y^2 = e^x(-x \sin y - \sin y - \sin y - y \cos y)$ -----(ii)

Adding equations i and ii ;

$$v_x^2 + v_y^2 = 0$$

Therefore, v satisfies Laplace equation and thus v is harmonic.

$v_x = e^x(x \sin y + y \cos y) + e^x \sin y$

$\Psi_1(z,0) = 0$

$v_y = e^x(x \cos y + \cos y - y \sin y)$

$\Psi_2(z,0) = e^z(z + 1)$

$\Rightarrow f(z) = \Psi_1(z,0) + i\Psi_2(z,0)$

$f(z) = \int e^z(z+1)\, dz$

$= ze^z$

Ans : $f(z) = ze^z$

Q3.a) From 8 observations the following results were obtained : [6]

$\Sigma x = 59; \Sigma y = 40; \Sigma x^2 = 524; \Sigma y^2 = 256; \Sigma xy = 364$

Find the equation of line of regression of x on y and the coefficient of correlation.

Sol : $\bar{X} = \frac{59}{8} = 7.375$; $\bar{Y} = \frac{40}{8} = 5$

Coefficient of regression of y on x :

$$\Rightarrow b_{yx} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}$$

$$\Rightarrow b_{yx} = \frac{364 - \frac{(59)(40)}{8}}{524 - \frac{(59)^2}{8}}$$

$\therefore b_{yx} = 0.7764$

Coefficient of regression of x on y :

$$\Rightarrow b_{xy} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma y^2 - \frac{(\Sigma y)^2}{N}}$$

$$\Rightarrow b_{xy} = \frac{364 - \frac{(59)(40)}{8}}{256 - \frac{(40)^2}{8}}$$

$\therefore b_{xy} = 1.2321$

Equation of line of regression of x on y is given by

$X - \bar{X} = b_{xy}(Y - \bar{Y})$

$\Rightarrow X - 7.375 = 1.2321(Y - 5)$

$\Rightarrow X = 1.2321(Y - 5) + 7.375$

$\Rightarrow X = 1.2321Y + 1.2145$

$r = \sqrt{b_{xy} . b_{yx}}$

$r = \sqrt{(1.2321)(0.7764)}$

r = 0.9781

Ans : X = 1.2321Y + 1.2145

r = 0.9781

Q3.b) Find the bilinear transformation which maps the points z=-1, 0, 1 onto the plane w=-1, -i, 1 **[6]**

Sol : Let z=-1, 0, 1 be the points in the z-plane with the images w=-1, -I, 1 in the w plane.

The bilinear transformation is given by,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(-1+i)(1-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(1-w)(-1+i)} = \frac{(z+1)}{(1-z)}$$

$$\Rightarrow \frac{-(w+1)(i+1)}{-(w-1)(-1+i)} = \frac{(z+1)}{-(z-1)} \qquad \Rightarrow \frac{(w+1)(i+1)}{-(w-1)(1-i)} = \frac{(z+1)}{-(z-1)}$$

$$\Rightarrow \frac{(w+1)(i+1)}{(w-1)(1-i)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)(i+1)}{(w-1)(1-i)}\frac{(1+i)}{(1+i)} = \frac{(z+1)}{(z-1)} \qquad \text{----- (i)} \qquad \text{(Rationalising)}$$

$$\Rightarrow \frac{(w+1)(i+1)^2}{(w-1)(1^2-i^2)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)(-1+2i+1)}{(w-1)(2)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)i}{(w-1)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)}{(w-1)} = \frac{(z+1)}{(iz-i)}$$

Applying componendo – dividendo;

$$\Rightarrow \frac{(w+1)+(w-1)}{(w+1)-(w-1)} = \frac{(z+1)+(iz-i)}{(z+1)-(iz-i)}$$

$$\Rightarrow \frac{2w}{2} = \frac{z+1+iz-i}{z+1-iz+i}$$

$$\Rightarrow w = \frac{z(1+i)+(1-i)}{z(1-i)+(1+i)}$$

$$\Rightarrow w = \frac{z\frac{(1+i)}{(1-i)}+1}{z+\frac{(1+i)}{(1-i)}}$$

From above steps(rationalising eqn i we know $(1 + i)/(1 - i) = i$)

$$\Rightarrow w = \frac{zi+1}{z+i}$$

Ans : Therefore, the required transformation, w = $\frac{zi+1}{z+i}$

Q3.c) Obtain half – range cosine series for f(x) **=** $(x-1)^2$ in $0 < x < 1$**.**

Hence find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ **[8]**

Sol : $f(x) = (x-1)^2$ in $0 < x < 1$

∴The half range cosine series or f(x) is given as :

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{1}\int_0^1 (x-1)^2 dx$$

$$\Rightarrow a_0 = 1\left[\frac{(x-1)^3}{3}\right]_0^1 \qquad \Rightarrow a_0 = 0 - (-\frac{1}{3}) \qquad \Rightarrow a_0 = \frac{1}{3}$$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x dx$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = 2\left[0+0-0-\left\{0-\frac{2}{n^2\pi^2}-0\right\}\right]$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

$$\therefore (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Put x=0;

$$\Rightarrow 1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{2}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Ans : $(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Q4.a) Find the inverse Laplace transform by using convolution theorem $\frac{1}{(s^2+a^2)(s^2+b^2)}$ **[6]**

Sol : $L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\sin at$

$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{(s^2+b^2)}\right] = \frac{1}{b}\sin bt$

$L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \frac{1}{a}\sin au.\frac{1}{b}\sin b(t-u)du$

$\Rightarrow \frac{1}{ab}\int_0^t \sin au.\sin b(t-u)du$

$$\Rightarrow \frac{-1}{2ab}\int_0^t \{\cos([a-b]u+bt)-\cos[a+b]u-bt)\}du$$

$$\left\{\because \sin A \sin B = -\frac{1}{2}[\cos(A+B)-\cos(A-B)]\right\}$$

$$\Rightarrow \frac{-1}{2ab}\left[\frac{\sin\{(a-b)u+bt\}}{a-b}-\frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$\Rightarrow \frac{-1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right]$$

$$\Rightarrow \frac{-1}{2ab}\left[2b.\frac{\sin at}{a^2-b^2}-2a.\frac{\sin bt}{a^2-b^2}\right]$$

$$\Rightarrow \left[\frac{a.\sin bt}{a^2-b^2}-\frac{b.\sin at}{a^2-b^2}\right]$$

Ans : $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \left[\frac{a.\sin bt}{a^2-b^2}-\frac{b.\sin at}{a^2-b^2}\right]$

Q4.b) Compute Spearman's Rank correlation coefficient for the following data: [6]

X	85	74	85	50	65	78	74	60	74	90
Y	78	91	78	58	60	72	80	55	68	70

Sol :

X	R1	Y	R2	D	D^2 $=(R1-R2)^2$
85	8.5	78	7.5	1	1
74	5	91	10	-5	25
85	8.5	78	7.5	1	1
50	1	58	2	-1	1
65	3	60	3	0	0
78	7	72	6	1	1
74	5	80	9	-4	16
60	2	55	1	1	1

74	5	68	4	1	1
90	10	70	5	5	25
N=10					∑=72

Therefore, $R = 1 - \dfrac{6\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_1^3 - m_1) + \ldots\}}{N^3 - N}$

Here m1=2, m2=2, m3=3,

$$R = 1 - \frac{6\{72 + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2) + \frac{1}{12}(3^3-3) + \ldots\}}{10^3 - 10}$$

On solving, R=0.5454

Ans : R=0.5454

Q4.c) Find the inverse Z-transform for the following: [8]

i) $\dfrac{1}{(z-5)^2}$, $|z| < 5$

ii) $\dfrac{z}{(z-2)(z-3)}$, $|z| > 3$

Sol :

i) $\dfrac{1}{(z-5)^2}$, $|z| < 5$

$$\Rightarrow \frac{1}{5^2\left(1-\left(\frac{5}{z}\right)\right)^2}$$

$$\Rightarrow \frac{1}{5^2}\left[1-\left(\frac{z}{5}\right)\right]^{-2}$$

$$\Rightarrow \frac{1}{5^2}\left[1+2\left(\frac{z}{5}\right)+3\left(\frac{z}{a}\right)^2+\ldots+(n+1)\left(\frac{z}{5}\right)^n\right]^1$$

$$\Rightarrow \left[\frac{1}{5^2}+2\left(\frac{z}{5^3}\right)+3\left(\frac{z^2}{5^4}\right)^2+\ldots+(n+1)\left(\frac{z^n}{5^{n+2}}\right)\right]^1$$

Coefficient of $z^n = \dfrac{n+1}{5^{n+2}}, n \geq 0$

Put n = -k ;

Coefficient of $z^{-k} = \frac{-k+1}{5^{-k+2}}, k \le 0$

Ans : $Z^{-1}[F(z)] = \frac{-k+1}{5^{-k+2}}, k \le 0$

ii) $\frac{z}{(z-2)(z-3)}, |z| > 3$

Applying Partial Fractions;

$\frac{z}{(z-2)(z-3)} = \frac{A}{z-3} + \frac{B}{z-2}$ ------ (i)

$\Rightarrow z = A(z-2) + B(z-3)$

Put z=2

$\Rightarrow 2 = B(-1) \Rightarrow$ **B = -2**

Put z=3

$\Rightarrow 3 = A(1) \Rightarrow$ **A = 3**

Resubstituting in (i);

$\frac{z}{(z-2)(z-3)} = \frac{3}{z-3} - \frac{2}{z-2}$

RHS:

$\Rightarrow -\frac{3}{3\left[1-\left(\frac{z}{3}\right)\right]} + \frac{2}{2\left[\left(1-\left(\frac{z}{2}\right)\right)\right]}$

$\Rightarrow \left(1-\frac{z}{2}\right)^{-1} - \left(1-\frac{z}{3}\right)^{-1}$

$\Rightarrow \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^{2}+...+\left(\frac{z}{2}\right)^{n}\right] - \left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^{2}+...+\left(\frac{z}{3}\right)^{n}\right)$

The coefficient of $z^{n} = 2^{-n} - 3^{-n}; n \ge 0$

Put n=-k;

$z^{-k} = 2^{k} - 3^{k}; k \le 0$

Ans : $Z^{-1}[F(z)] = 2^{k} - 3^{k}, k \le 0$

Q5.a) Using Laplace Transform evaluate $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt$ **[6]**

Sol : To evaluate $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt$

$\Rightarrow f(t) = 1 + 3t + t^2$; a=2

$\Rightarrow f(t+1) = 1 + 3(t+2) + (t+2)^2$

$= 1 + 3t + 6 + (t^2 + 4t + 4)$

$= t^2 + 7t + 11$

$L[f(t+2)] = L[t^2 + 7t + 11]$

$= \frac{2!}{s^3} + 7\frac{1!}{s^2} + \frac{11}{s}$ ----- i

We know, $L[f(t)H(t-a)] = e^{-as}L[f(t + a)]$

Substituting the value of L[f(t+a)] in above equation, we get

$$L[(1+3t+t^2)H(t-2)] = e^{-2s}\left[\frac{2!}{s^3}+7\frac{1!}{s^2}+\frac{11}{s}\right]$$

$$\int_0^\infty e^{-st}(1+3t+t^2)H(t-2)dt = e^{-2s}\left[\frac{2!}{s^3}+7\frac{1!}{s^2}+\frac{11}{s}\right]$$

Putting s=1 in the above equation;

$$\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt = e^{-2}\left[\frac{2!}{1}+7\frac{1!}{1}+\frac{11}{1}\right]$$

$$=e^{-2}[2+7+11] = \frac{20}{e^2}$$

Ans : $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt = \frac{20}{e^2}$

Q5.b) Prove that $f_1(x) = 1; f_2(x) = x; f_3(x) = \frac{3x^2-1}{2}$ are orthogonal over (-1,1).

[6]

Sol : Conditions for functions to be orthogonal are

i) $\int_a^b f_m(x).f_n(x)dx=0$; $m \neq n$

ii) $\int_a^b [f_n(x)]^2 dx \neq 0$; $m=n$

i) Proving 1st condition is true,

We have, $\int_{-1}^{1} f_1(x).f_2(x)dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1}$

$\Rightarrow \frac{1}{2}(1^2 - (-1)^2) = 0$

$\int_{-1}^{1} f_1(x).f_3(x)dx = \int_{-1}^{1} \frac{3x^2-1}{2}\,dx = \frac{1}{2}[x^3-x]_{-1}^{1}$

$\Rightarrow \frac{1}{2}[(1^3-1)-\{(-1)^3-(-1)\}] \Rightarrow \frac{1}{2}[(0)-(0)] = 0$

$\int_{-1}^{1} f_2(x).f_3(x)dx = \int_{-1}^{1} \frac{x}{2}(3x^2-1)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3-x)\,dx$

$\Rightarrow \frac{1}{2}[\frac{3x^4}{4} - \frac{x^2}{2}]_{-1}^{1} \Rightarrow \frac{1}{2}[\left(\frac{3}{4}-\frac{1}{2}\right) - (\frac{3}{4} - \frac{1}{2})] = 0$

ii) Proving 2nd condition in true;

$\int_{-1}^{1} [f_1(x)]^2 dx = \int_{-1}^{1} 1^2 dx = [x]_{-1}^{1} = [1-(-1)] = 2 \quad \neq 0$

$\int_{-1}^{1} [f_2(x)]^2 dx = \int_{-1}^{1} x^2 dx = [\frac{x^3}{3}]_{-1}^{1} = \left[\frac{1}{3}-\left(-\frac{1}{3}\right)\right] = \frac{2}{3} \quad \neq 0$

$\int_{-1}^{1} [f_3(x)]^2 dx = \int_{-1}^{1} (\frac{3x^2-1}{2})^2\,dx$

$\Rightarrow \frac{1}{4}\int_{-1}^{1}(9x^4-6x^2+1)dx \Rightarrow \frac{1}{4}\left[\frac{9x^5}{5}-2x^3+x\right]_{-1}^{1}$

$\Rightarrow \frac{1}{4}\left[\frac{9}{5}-2+1-\left\{-\frac{9}{5}-2(-1)^3-1\right\}\right]$

$\Rightarrow \frac{1}{4}\left[\frac{18}{5}-4+2\right] \Rightarrow \frac{2}{5} \neq 0$

Hence, the given set is orthogonal on [-1,1]

Q5.c) Solve using Laplace transform

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}\text{ ; } y = 2 \text{ and } y' = 3 \text{ at } x = 0 \qquad [8]$$

Sol : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$

$\therefore (D^2 - 3D + 2)y = 2e^{3x}$

$\therefore y'' - 3y' + 2y = 2e^{3x}$

Taking Laplace transform on both sides, we get

$L[y''] - 3L[y'] + 2L[y] = \frac{2}{s-3}$ $\qquad \{\because L[e^{at}] = \frac{1}{s-a}\}$

$L[y''] = s^2\dot{y} - sy(0) - y'(0)$

$L[y'] = s\dot{y} - y(0)$

Substituting the values in the equation,

$$s^2\dot{y} - 2s - 3 - 3(s\dot{y} - 2) + 2\dot{y} = \frac{2}{s-3}$$

$\Rightarrow \dot{y}(s^2-3s+2) - 2s + 3 = \frac{2}{s-3}$

$$\Rightarrow \dot{y}(s^2-3s+2) = \frac{2}{s-3} + (2s-3)$$

$$\Rightarrow \dot{y}(s^2-3s+2) = \frac{2+(2s-3)(s-3)}{s-3}$$

$$\Rightarrow \dot{y}(s^2-3s+2) = \frac{2s^2-9s+11}{s-3}$$

$$\Rightarrow \dot{y} = \frac{2s^2-9s+11}{(s^2-3s+2)(s-3)}$$

$\Rightarrow \dot{y} = \frac{2s^2-9s+11}{(s-1)(s-2)(s-3)}$ $\qquad [(x^2-3x+3)=(x-1)(x-2)]$

Applying partial fractions;

$$\frac{2s^2-9s+11}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$\Rightarrow 2s^2 - 9s + 11 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

Put s=1 $\qquad$ Put s=2 $\qquad$ Put s=3

$4 = 2A$ $\qquad$ $1 = -B$ $\qquad$ $2 = 2C$

A=2 B=-1 C=1

$$\therefore \frac{2s^2-9s+11}{(s-1)(s-2)(s-3)} = \frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}$$

$$\bar{y} = \frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}$$

Taking inverse Laplace on both sides,

$$L^{-1}\left[\bar{y}\right] = L^{-1}\left[\frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}\right]$$

$$y = 2e^{t} - e^{2t} + e^{3t}$$

Ans : $y = 2e^{t} - e^{2t} + e^{3t}$

Q6.a) Find the complex form of the Fourier series for $f(x) = e^{x}, (-\pi, \pi)$ [6]

Sol : The complex form of the Fourier series for $f(x) = e^{x}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx} \text{ where } C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}.e^{-inx} dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x} dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{(1-in)}\right]_{-\pi}^{\pi}$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\left\{\frac{e^{(1-in)\pi}}{(1-in)}\right\} - \left\{\frac{e^{(1-in)(-\pi)}}{(1-in)}\right\}\right]$$

$$\Rightarrow C_n = \frac{1}{2\pi(1-in)}\left[e^{\pi}.e^{-in\pi} - e^{-\pi}e^{in\pi}\right]$$

But $e^{\pm(in\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$\therefore C_n = \frac{1}{2\pi(1-in)}\left[e^{\pi}.(-1)^{n} - e^{-\pi}(-1)^{n}\right]$$

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\sinh \pi \qquad \left\{\because \frac{e^{x}-e^{-x}}{2} = \sinh(x)\right\}$$

Rationalising the denominator, multiply divide by (1+in);

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\sinh\pi \cdot \frac{1+in}{1+in}$$

$$\Rightarrow C_n = \frac{(-1)^n(1+in)}{\pi(1^2-(in)^2)}\sinh\pi \Rightarrow \frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi$$

Substituting the value in f(x)

$$f(x) = \sum_{-\infty}^{\infty}\frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi . e^{inx}$$

Ans : $e^x = \sum_{-\infty}^{\infty}\frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi . e^{inx}$

Q6.b) If u, v are harmonic conjugate functions, show that uv is a harmonic function [6]

Sol : Let f(z) = u + iv be the analytic function;

$\therefore u_x = v_y$ and $u_y = -v_x$

And u, v are harmonic therefore $u_x^{2} + u_y^{2} = 0$ and $v_x^{2} + v_y^{2} = 0$ -----(i)

Now, $uv_x = uv_x + vu_x$

$(uv)_x^{2} = u_xv_x + u(v_x)^2 + v_xu_x + v(u_x)^2$

$(uv)_x^{2} = 2u_xv_x + u(v_x)^2 + v(u_x)^2$ -----(ii)

Similarly, we can prove that,

$(uv)_y^{2} = 2u_yv_y + u(v_y)^2 + v(u_y)^2$

But $u_x = v_y$ and $u_y = -v_x$

$\therefore (uv)_y^{2} = -2u_xv_x + u(v_y)^2 + v(u_y)^2$ -----(iii)

Adding (ii) and (iii), we get;

$(uv)_x^{2} + (uv)_y^{2} = u\left(v_x^{2}+v_y^{2}\right) + v(u_x^{2} + u_y^{2})$

$=0$ {from i}

Therefore, uv is harmonic

Q6.c) Fit a straight line of the form, y = a + bx to the following data and estimate the value of y for x = 3.5 [8]

X	0	1	2	3	4
Y	1	1.8	3.3	4.5	6.3

Solution:-

x	y	x^2	xy
0	1.0	0	0.0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum$=10	$\sum$=16.9	$\sum$=30	$\sum$=47.1

Here N=5.

Let the equation of the line be y = a + bx

Then the normal equations are :

$\sum y = Na + b\sum x$

$\sum xy = N\sum x + b\sum x^2$

Substituting the values in the above equation,

$\therefore 16.9=5a+10b$

$\therefore 47.1=10a+30b$

Solving the above equations simultaneously,

a = 0.72 and b= 1.33

$y = 0.72 + 1.33x$

At x=3.5; substituting the value in above equation,

$y = 0.72 + 1.33(3.5)$

$y = 5.375$

Ans : y = 0.72 + 1.33(x)

y at x = 3.5 :5.375
