Paper / Subject Code: 24291 / Mathematics: Topology of Metric Spaces (R 2022)

Duration 2 $\frac{1}{2}$Hrs **NEW COURSE** **Marks: 75**

N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) Define an open ball $B(x,r)$ in a metric space (X,d) and show that every open ball is an open set. Also give an example to show that the converse need not be true.
   (ii) Show that for a subset F of a metric space (X,d), the following statements are equivalent:
   (I) F is closed
   (II) F contains all its limit points.

   (b) Attempt any Two from the following: (12)
   (i) State and prove Hausdorff property in a metric space (X,d).
   (ii) Prove that $(\mathbb{N},d)$ and $(\mathbb{N},d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{N}$, are equivalent metric spaces.
   (iii) Let $X=\mathbb{R}^2$ and d be the Euclidean metric on X. Show that $A=\{(x_1,x_2)\in\mathbb{R}^2 : 0< x_1<1, 0<x_2<1\}$ is open in X.

2. (a) Attempt any One from the following: (8)
   (i) If in a metric space (X,d), for every decreasing sequence $\{F_n\}$ of non-empty closed sets with diam $F_n \longrightarrow 0$, we have $\cap F_n$ is a singleton set then prove that (X,d) is complete.
   (ii) Let (X,d) be a metric space and A be a subset of X. Show that $p\in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p.

   (b) Attempt any Two from the following: (12)
   (i) Show that a sequence (x_n) in $(\mathbb{R}^2,d)$ (where d is Euclidean distance) converges to a point $p=(p_1,p_2)\in\mathbb{R}^2$ if and only if $(x_n{}^i)\longrightarrow p_i$ for $1\le i\le 2$, in $\mathbb{R}$ with respect to the usual distance, where $x_n=(x_n{}^1,x_n{}^2)$.
   (ii) Let (X,d) be a metric space and (x_n) be a Cauchy sequence in X. If (x_n) has a convergent subsequence then prove that sequence (x_n) itself is convergent.
   (iii) Check whether Cantor's Theorem is applicable in each of the following examples and find $\bigcap_{n\in\mathbb{N}}F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance d is usual distance from $\mathbb{R}$, in each examples:
   (i) $X=[-1,1]$, $F_n=[-\frac{1}{n},\frac{1}{n}]$
   (ii) $X=(0,1)$, $F_n=(0,\frac{1}{n+1}]$

3. (a) Attempt any One from the following: (8)
   (i) For a nonempty subset A of metric space $(\mathbb{R},d)$, where d is usual metric, prove that if A is closed and bounded then it satisfies Hein-Borel property.
   (ii) If $\mathcal{C}$ is a non-empty collection of compact subsets of a metric space (X,d) then prove that $\bigcap_{K\in\mathcal{C}} K$ is a compact subset of X. Further, if $\mathcal{C}$ is finite then show that $\bigcup_{K\in\mathcal{C}} K$ is a compact subset of X.

(b) Attempt any Two from the following: (12)

(i) Prove or disprove: $A = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ is a compact subset of $(\mathbb{R}^2, d)$, where d is Euclidean distance.

(ii) If A, B are compact subsets of $\mathbb{R}$ with respect to usual distance then prove that $A + B$ is a compact subset of $\mathbb{R}$ with usual metric.

(iii) Prove that the open cover $\{B(0, n)\}_{n\in\mathbb{N}}$ of a metric space $(C[a, b], \| \|_\infty)$, where $\|f\|_\infty = \sup \{|f(t)| : t \in [a, b]\}$, has no finite subcover. (0 being the constant zero function)

4. Attempt any Three from the following: (15)

(a) Define an open set in a metric space (X, d). Let $A \subseteq X$. Show that A is open if and only if $A = A^\circ$ (Interior of A).

(b) Let (X, d) be a metric space. If A is any finite non-empty subset of X then show that $X \setminus A$ is an open set.

(c) Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = (x - a)^2(x - b)^2 + x$, takes the value $\frac{a+b}{2}$ for some value of $x \in \mathbb{R}$. (distance in $\mathbb{R}$ being usual)

(d) Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.

(e) Let (X, d) be a metric space and $(x_n) \in X$ such that $(x_n) \longrightarrow p$ in X. Show that $K = \{x_n ; n \in \mathbb{N}\} \cup \{p\}$ is a compact subset of X using the definition.

(f) Show that the set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is a closed and bounded subset in $\mathbb{Q}$ with usual metric, but not compact.

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