(3 Hours) [Total Marks : 100

N.B.: 1. All questions are compulsory.
2. Figures to the right indicate full marks.

Q.1 Choose the correct alternative in each of the following: (20)

i. $\lim_{Z\to 0}\frac{\bar{z}}{z} =$
(a) 1 (b) i (c) $-i$ (d) does not exist

ii. The image of a line under a fractional linear transformation is
(a) a line (b) a circle (c) a line or a circle (d) None of these

iii. $f(z) = \frac{z^2+1}{z^3+9}$ is
(a) continuous and bounded in $|z| \le 2$
(b) continuous but not bounded in $|z| \le 2$
(c) neither continuous nor bounded in $|z| \le 2$
(d) continuous and bounded everywhere

iv. If $u(x,y) = x^2 - y^2, v = 2xy$ then
(a) v and u are harmonic conjugates of each other
(b) u is a harmonic conjugate of v but v is not a harmonic conjugate of u
(c) v is a harmonic conjugate of u but u is not a harmonic conjugate of v
(d) None of these

v. $\exp(2 \pm 3\pi i) =$
(a) e^{-2} (b) $-e^2$ (c) e^2 (d) e^3

vi. If $e^z = -2$, then $z =$
(a) 0 (b) $z = ln\,2 + (2n+1)\pi i \quad n = 0, \pm 1, \pm 2, \ldots$
(c) i (d) none of these

vii. $\int_C \frac{e^z}{z-2}\,dz$, where C is the circle $|z| = 3$, described in the positive sense is
(a) $2\pi i\ e^2$ (b) $2\pi i$ (c) e^2 (d) None of these

viii. Radius of convergence of the series $\sum_{n=0}^{\infty}\frac{(z-2i)^n}{n^n}$ is
(a) ∞ (b) 1 (c) 2 (d) None of these

ix. The poles of the function $\frac{\sin z}{\cos z}$ are at
(a) $\frac{(2n+1)\pi}{2}$, n is any integer (b) $\frac{2n\pi}{3}$, n is any integer
(c) $n\pi$, n is any integer (d) none of these

x. The residue of f at $z = 0$ where $f(z) = z\cos\frac{1}{z}$ is

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) none of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Let $f(z) = u(x, y) + iv(x, y)$. If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then prove that the first order partial derivatives of u and v exist at (x_0, y_0) and satisfy Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. Also show that $f'(z) = (u_x)_{z=z_0} + i(v_x)_{z=z_0}$.

ii. If $f'(z_0)$, $g'(f(z_0))$ exist then prove that the function $F(z) = g(f(z))$ has a derivative at z_0 and $F'(z_0) = g'(f(z_0))f'(z_0)$. If $f: A \subseteq \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in A$, then show that f is continuous at z_0. Let $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ such that f is differentiable at $z_0 \in \Omega$, then show that $\exists$ a function $\eta(z)$ such that $f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0)$ where $\eta(z) \to 0$ as $z \to z_0$.

b) Attempt any TWO questions from the following: (12)

i. Show that $z(t) = z_0 + tv$ and $Re((z - z_0)i\,\bar{v}) = 0$ represents the same line in $\mathbb{C}$.

ii. If $f'(z) = 0$ everywhere on a domain D then show that $f(z)$ must be constant throughout D.

iii. Show that $f(z) = z\,|z|$ is differentiable everywhere when f is treated as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ but $\mathbb{C}$ differentiable only at $z = 0$.

iv. If $f(z) = 8x - x^3 - xy^2 + i(x^2y + y^3 - 8y)$ then determine points at which f is differentiable, f is analytic.

Q.3 a) Attempt any ONE question from the following: (08)

i. f is a analytic inside and on a simple, closed curve C, taken in the positive sense. Prove that $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2}$. Further state the result generalizing the formula to $f^n(z)$.

ii. Define complex sine and cosine functions. Also establish the following three identities:

$e^z \neq 0 \;\forall\, z \in \mathbb{C}$

$sin^2 z + cos^2 z = 1$

$|\sinh z|^2 = \sinh^2 x + sin^2 y$

b) Attempt any TWO questions from the following: (12)

ii. Evaluate the integral $\int_C \frac{z+3}{z^2-5z+6}\,dz$, where

(I) $C: |z - 2| = \frac{1}{2}$ (II) $C: |z - 3| = \frac{3}{4}$

iii. Find a Mobius transformation that maps $i, \infty, 3$ to $1/2, -1, 3$ respectively.

iv. State Taylor's theorem and also find Taylor series for $f(z) = \frac{e^z}{1-z}$ around $z = 0$.

Q.4 a) Attempt any ONE question from the following: (08)

i. If C is a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z)$ be any function which is continuous on C then prove that the series $\sum_{n=0}^{\infty} g(z) a_n (z - z_0)^n$ can be integrated term by term over C and $\int_C g(z) S(z) dz = \sum_{n=0}^{\infty} \int_C g(z) a_n (z - z_0)^n dz$.

ii. If z_1 is a point inside the circle of convergence $|z - z_0| = R$ of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ then show that the series must be uniformly convergent in the closed disk $|z - z_0| \le R_1$, where $R_1 = |z_1 - z_0|$.

b) Attempt any TWO questions from the following: (12)

i. Define the following terms: A removable singularity, A pole of order n, An essential singularity.

ii. With the help of a suitable power series, show that $f(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$ is entire, use it to show that $\lim_{z \to 0} \frac{e^z - 1}{z} = 1$.

iii. Write Laurent series representations of function $f(z) = \frac{1}{z(4-z)^2}$ in the domains $|z| < 4$ and $|z| > 4$.

iv. Evaluate the real improper integral $\int_0^{\infty} \frac{dx}{x^2+1}$ using the method of residue.

Q.5 Attempt any FOUR questions from the following: (20)

a) Use Cauchy Riemann equations to check differentiability of $f(z) = Re\ z$

b) Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when $u(x, y) = x^3 - 3xy^2$

c) Find image of the set $\{z \in \mathbb{C} \,/\, |z| = \frac{1}{4},\ \pi/2 \le \arg(z) \le \pi\}$ under the reciprocal map $w = 1/z$ on the extended complex plane.

d) Find values of z such that $exp(2z - 1) = 1$.

e) Let $\sum a_n z^n$ has radius of convergence R. Find the radius of convergence of $\sum_{n=0}^{\infty} n^3 a_n z^n$, $\sum_{n=0}^{\infty} a_n z^{3n}$

f) Evaluate $\int_C \frac{dz}{(z^2+1)(z^2-4)}$, where C is circle $|z| = 1$.

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